Optimum Asymptotic Multiuser Efficiency of Pseudo-Orthogonal Randomly Spread CDMA

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Abstract

A K-user pseudo-orthogonal (PO) randomly spread CDMA system, equivalent to transmission over a subset of $K' \leq K$ single-user Gaussian channels, is introduced. The high signal-to-noise ratio performance of the PO-CDMA is analyzed by rigorously deriving its asymptotic multiuser efficiency (AME) in the large system limit. Interestingly, the K'-optimized PO-CDMA transceiver scheme yields an AME which is practically equal to 1 for system loads smaller than 0.1 and lower bounded by 1/4 for increasing loads. As opposed to the vanishing efficiency of linear multiuser detectors, the derived efficiency is comparable to the ultimate CDMA efficiency achieved for the intractable optimal multiuser detector.

1 Introduction

Non-orthogonality of realistic spreading sequences, serving as signature codes in direct-sequence code division multiple-access (DS-CDMA), has an inherent detrimental effect in cellular communications. Nevertheless, a de-facto orthogonality between transmissions can be achieved via a proper joint binary signaling, for which multiple-access interference (MAI) agrees with information polarity.

By borrowing analysis tools from statistical mechanics, counting the number of metastable states of the Hopfield model of neural networks [1], we rigorously derive the asymptotic multiuser efficiency (AME, [2]) of this pseudo-orthogonal modulation scheme for the common random spreading CDMA channel and draw its optimum value.

2 Pseudo-Orthogonal Random CDMA

Consider a perfectly power-controlled, synchronous, K-users, N-chips random binary spreading, binary signaling, Gaussian CDMA channel. A random CDMA channel is termed 'Pseudo-Orthogonal' (PO) if the transmitted signaling vectors $\{-1,1\}^K \ni \mathbf{b} \triangleq \{b_1,\ldots,b_K\}$ are chosen such that

$$\mathbf{R}\mathbf{b} \equiv \mathbf{D}\mathbf{b}, \tag{1}$$

or in scalar form

$$\sum_{i=1}^{K} \rho_{ki} b_i \equiv \lambda_k b_k \quad \forall k = 1\ldots K, \tag{2}$$

where the $K \times K$ symmetric matrix $\mathbf{R}$, with entries ρ_{ki}, is the spreading sequences' normalized crosscorrelation matrix, and the $K \times K$ matrix

$$\mathbf{D} \triangleq \mathrm{diag}\{\lambda_1 > 0, \ldots, \lambda_{K'} > 0, \lambda_{K'+1} > -\infty, \ldots, \lambda_K > -\infty\} \tag{3}$$

is a diagonal matrix, in which the first $K' \leq K$ diagonal entries must be positive and the rest can get any value.

To put differently, the allowed signaling vectors $\mathbf{b}$ are those for which the *MAI does not flip the information bits* of a given fraction $\gamma \triangleq K'/K$ of the users. Evidently, this pseudo-orthogonality is gained at the cost of reducing the signaling entropy per user, H, to be less than unity (as opposed to multiuser precoding [3] in which there is no restriction on the input signaling.)

On the other hand, this scheme reverts the multiuser channel to an equivalent set of γH single-user additive white Gaussian noise (AWGN) channels, with much simpler detectors w.r.t. the intractable optimal multiuser detector (MUD). This joint signaling scheme is particularly attractive for the CDMA downlink (but may also be utilized for the uplink in case of cooperation between users.) In order to evaluate the high signal-to-noise ratio (SNR) bit-error performance of the PO-CDMA scheme, its AME is analyzed.

3 Asymptotic Multiuser Efficiency

Theorem 1 *In the large-system limit analysis,* i.e., *$K, N \to \infty$, yet the system load factor $\beta \triangleq K/N$ is kept constant, the PO-CDMA channel's AME, which characterizes the performance loss (in effective SNR) as the background noise vanishes [2], is proved (for a given active users fraction γ) to get the form*

$$\begin{aligned}\eta(\beta,\gamma) &= \frac{\gamma \log_2 e}{\beta}(b^* - \frac{1}{2} + \frac{(1-b^*)^2}{2a^*} + \frac{1}{2}\log a^*) \\ &+ \gamma^2 \log_2\left(2Q(t^*)\right) + \gamma(1-\gamma),\end{aligned}$$

where $Q(\cdot)$ is the complementary cumulative distribution function of a standard Gaussian random variable[1]*, and the fixed-points a^*, b^* and t^* are devised by solving numerically the following set of equations*

$$\begin{aligned}\beta^{-1}(\frac{(1-b)^2}{a} - 1) + \gamma t\frac{Q'(t)}{Q(t)} &= 0, \\ \beta^{-1}(1 - \frac{1-b}{a}) + \frac{\gamma}{\sqrt{a\beta}}\frac{Q'(t)}{Q(t)} &= 0,\end{aligned}$$

with an auxiliary variable $t \triangleq (b-1)/\sqrt{a\beta}$.

4 Proof

If a particular multiple-access system achieves bit-error-rate (BER) per user $P(\sigma)$ in the presence of MAI and AWGN with power spectral level equal to σ^2, then the AME w.r.t. a single-user system (or a fully-orthogonal CDMA) is defined by [2]

$$\eta \triangleq \lim_{\sigma \to 0} e(\sigma). \tag{4}$$

[1] $Q(x) \triangleq 1/\sqrt{2\pi}\int_x^\infty dy \exp\left(-y^2/2\right)$

The term $e(\sigma)$ is the energy per user required to achieve BER equal to $P(\sigma)$ in a single-user Gaussian channel with the same background noise level.

According to definition (4), the AME for the PO-CDMA case (1), $\eta(\beta,\gamma)$, is nothing but $\gamma H(\beta,\gamma)$, the fraction of 'orthogonal' users multiplied by their signaling entropy (as one is interested in an equivalent set of single-user channels with unit entropy.) Thus we have to compute the non-trivial entropy $H(\beta,\gamma)$ under the PO-CDMA transmission constraints (2).

A K-length binary transmission codeword $\mathbf{b}^c \triangleq \{b_1^c,\ldots,b_K^c\}$, composed of all K users' bits at a given channel use, for which the PO constraints (2) hold, satisfies the condition

$$\int_0^\infty \prod_{k=1}^{K'} d\lambda_k \delta(\sum_i \rho_{ki} b_i - \lambda_k b_k) = \int_{-1}^\infty \prod_{k=1}^{K'} d\lambda_k \delta(\sum_{i\neq k} \rho_{ki} b_i - \lambda_k b_k) = 1, \qquad (5)$$

where the function $\delta(\cdot)$ is the Dirac delta function. Condition (5) can be reformulated as

$$\alpha \int_{-1}^\infty \prod_{k=1}^{K'} d\lambda_k \delta(\sum_{i\neq k} \alpha\rho_{ki} b_i - \alpha\lambda_k b_k) =$$
$$\int_{-\alpha}^\infty \prod_{k=1}^{K'} d\lambda_k \delta(\sum_{i\neq k} \alpha\rho_{ki} b_i - \lambda_k b_k) = 1, \qquad (6)$$

where $\alpha \triangleq 1/\beta$.

Let the random variable $\mathbb{N}(\beta,\gamma,\mathbf{R})$ denote the number of PO-CDMA allowable codewords, *i.e.*

$$\mathbb{N}(\beta,\gamma,\mathbf{R}) \triangleq \int_{-\alpha}^\infty \prod_{k=1}^{K'} d\lambda_k \delta(\sum_{i\neq k} \alpha\rho_{ki} b_i - \lambda_k b_k). \qquad (7)$$

Assuming equal user information rates, the corresponding asymptotic signaling entropy is defined [4], in bit information units, as

$$H(\beta,\gamma) \triangleq \lim_{K\to\infty} \frac{\log_2 \mathbb{N}(\beta,\gamma,\mathbf{R})}{K}. \qquad (8)$$

Assuming self-averaging property [5], in the large-system limit $K\to\infty$ the number of successful codewords $\mathbb{N}(\beta,\gamma,\mathbf{R})$ is equal to its expectation w.r.t. the distribution of $\mathbf{R}$, *i.e.*

$$\lim_{K\to\infty} \mathbb{N}(\beta,\gamma,\mathbf{R}) = \mathcal{N}(\beta,\gamma)$$
$$= \lim_{K\to\infty} \int_{-\alpha}^\infty \prod_{k=1}^{K'} d\lambda_k \sum_{\mathbf{b}} \Big\langle \prod_{k=1}^{K'} \delta\Big(\sum_{i\neq k} \alpha\rho_{ki} b_i - \lambda_k b_k\Big)\Big\rangle_{\mathbf{R}}, \qquad (9)$$

where $\mathcal{N}(\beta,\gamma)$ and $< \cdot >_{\mathbf{R}}$ denote the average and averaging operation w.r.t. $\mathbf{R}$, respectively, and $\sum_{\mathbf{b}}$ corresponds to a sum over all the 2^K possible values of $\mathbf{b}$.

Representing the delta function by the inverse Fourier transform of an exponent, expression (9) can be rewritten as

$$\mathcal{N}(\beta,\gamma) = \lim_{K\to\infty} \int_{-\alpha}^\infty \prod_{k=1}^{K'} d\lambda_k \frac{1}{(2\pi)^{K'}} \int_{-\infty}^\infty \prod_{k=1}^{K'} d\omega_k$$

$$\times \quad \sum_{\mathbf{b}} \exp\Big(j\sum_{k=1}^{K'}\omega_k\lambda_k b_k\Big)\Big\langle \exp\Big(-j\sum_{i\neq k}\alpha\rho_{ki}b_i\omega_k\Big)\Big\rangle_{\mathbf{R}}, \tag{10}$$

where $j \triangleq \sqrt{-1}$ and ω is the angular frequency of the Fourier transform. Substituting $b_k\omega_k$ for ω_k, we find

$$\begin{aligned}\mathcal{N}(\beta,\gamma) &= \lim_{K\to\infty}\int_{-\alpha}^{\infty}\prod_k d\lambda_k \frac{1}{(2\pi)^K}\int_{-\infty}^{\infty}\prod_k d\omega_k \\ &\times \sum_{\mathbf{b}}\exp\Big(j\sum_k \omega_k\lambda_k\Big)\cdot\mathbb{E},\end{aligned} \tag{11}$$

where

$$\begin{aligned}\mathbb{E} &\triangleq \Big\langle \exp\Big(-j\sum_{i\neq k}\alpha\rho_{ki}b_i b_k\omega_k\Big)\Big\rangle_{\mathbf{R}} \\ &= \Big\langle \exp\Big(-j\sum_{i\neq k}\frac{1}{K}\sum_{\mu=1}^{N}s_k^\mu s_i^\mu b_i b_k\omega_k\Big)\Big\rangle_{\mathbf{R}}.\end{aligned} \tag{12}$$

In the last equality the cross-correlations $\rho_{ki} \triangleq 1/N\sum_{\mu=1}^{N}s_k^\mu s_i^\mu$ are expressed explicitly as a function of the binary chips $s_k^\mu = \pm 1$. The expectation $\mathbb{E}$ can be also written as

$$\mathbb{E} = \exp(j\alpha\sum_k\omega_k)\Big\langle \exp\Big(-\frac{j}{K}\sum_\mu(\sum_k s_k^\mu b_k\omega_k)(\sum_k s_k^\mu b_k)\Big)\Big\rangle_{\mathbf{R}}. \tag{13}$$

Using the transformation [6]

$$\begin{aligned}\exp(-jA_\mu B_\mu/K) &= \int_{-\infty}^{\infty}\frac{da_\mu}{(2\pi/K)^{1/2}}\int_{-\infty}^{\infty}\frac{db_\mu}{(2\pi/K)^{1/2}} \\ &\times \exp\Big(j\frac{K}{2}(a_\mu^2-b_\mu^2)-\frac{j}{\sqrt{2}}A_\mu(a_\mu+b_\mu)-\frac{j}{\sqrt{2}}B_\mu(a_\mu-b_\mu)\Big),\end{aligned} \tag{14}$$

expression (13) becomes (here, and hereafter, logarithms are taken to base e)

$$\begin{aligned}\mathbb{E} &= \exp(j\alpha\sum_k\omega_k)\int_{-\infty}^{\infty}\prod_\mu\frac{da_\mu}{(2\pi/K)^{1/2}}\int_{-\infty}^{\infty}\prod_\mu\frac{db_\mu}{(2\pi/K)^{1/2}} \\ &\times \exp\Big(j\frac{K}{2}\sum_\mu(a_\mu^2-b_\mu^2)+\sum_{k,\mu}\log(\cos(c_{k,\mu}))\Big),\end{aligned} \tag{15}$$

where

$$c_{k,\mu} \triangleq \frac{1}{\sqrt{2}}(\omega_k(a_\mu+b_\mu)+(a_\mu-b_\mu)). \tag{16}$$

Since $\sum_k s_k^\mu b_k$ in (13) is $\mathcal{O}(\sqrt{K})$ for a vast majority of codewords, for the expectation $\mathbb{E}$ to be finite, a_μ and b_μ must be $\mathcal{O}(1/\sqrt{K})$. Hence, expanding the $\log(\cos(\cdot))$ term in exponent (15) and neglecting terms of order $1/K$ and higher, we get

$$\begin{aligned}\mathbb{E} &= \exp(j\alpha\sum_k\omega_k)\int_{-\infty}^{\infty}\prod_\mu\frac{da_\mu}{(2\pi/K)^{1/2}}\int_{-\infty}^{\infty}\prod_\mu\frac{db_\mu}{(2\pi/K)^{1/2}} \\ &\times \exp\Big(j\frac{K}{2}\sum_\mu(a_\mu^2-b_\mu^2)-\frac{1}{4}\sum_{k,\mu}\hat{c}_{k,\mu}\Big),\end{aligned} \tag{17}$$

where

$$\hat{c}_{k,\mu} \triangleq (\omega_k^2(a_\mu + b_\mu)^2 + 2\omega_k(a_\mu^2 - b_\mu^2) + (a_\mu - b_\mu)^2). \tag{18}$$

Now, the solution of the K-dimensional integral (17) of the expectation $\mathbb{E}$ is performed using the following mathematical recipe: New variables are introduced

$$a \triangleq \frac{1}{2\alpha}\sum_\mu (a_\mu + b_\mu)^2, \tag{19}$$

$$b \triangleq \frac{j}{2\alpha}\sum_\mu (a_\mu^2 - b_\mu^2) + 1. \tag{20}$$

Equations (19) and (20) can be reformulated via the integral representation of a delta function using the corresponding angular frequencies A and B, respectively,

$$\int_{-\infty}^{\infty} \frac{da\, dA}{2\pi/K\alpha} \exp\left(jKA(\alpha a - \sum_\mu \frac{(a_\mu + b_\mu)^2}{2})\right) = 1, \tag{21}$$

$$\int_{-\infty}^{\infty} \frac{db\, dB}{2\pi/K\alpha} \exp\left(jKB(\alpha b - j\sum_\mu \frac{(a_\mu^2 - b_\mu^2)}{2} - \alpha)\right) = 1. \tag{22}$$

Substituting these (unity) integrals into the expectation expression (17) and rewriting it using a and b, the integrations over a_μ and b_μ are decoupled and can be performed easily. Next, for the asymptotics $K \to \infty$, the integration over the frequencies A and B can be performed algebraically by the saddle-point method [5].

According to this method, the main contribution to the integral comes from values of A and B in the vicinity of the maximum of the exponent's argument. Finally, the $\mathbb{E}$ term boils down to

$$\begin{aligned} \mathbb{E} &= \int_{-\infty}^{\infty} \frac{da\ db}{4\pi/K\alpha} \exp\left(-\frac{1}{2}\alpha a \sum_k \omega_k^2 + j\alpha b \sum_k \omega_k\right) \\ &\times \exp\left(K\alpha(b - \frac{1}{2} + \frac{(1-b)^2}{2a} + \frac{1}{2}\log a)\right). \end{aligned} \tag{23}$$

Substituting the expectation term (23) back in (11), the integrand in the latter becomes independent of $\mathbf{b}$, and therefore the $\sum_{\mathbf{b}}$ can be substituted by multiplying with the scalar 2^K. Hence,

$$\begin{aligned} \mathcal{N}(\beta,\gamma) &= \lim_{K\to\infty} \int_{-\alpha}^{\infty} \prod_k d\lambda_k \frac{2^{K-K'}}{\pi^{K'}} \int_{-\infty}^{\infty} \prod_k d\omega_k \exp\Big(j\sum_k \omega_k\lambda_k\Big) \\ &\times \int_{-\infty}^{\infty} \frac{da\ db}{4\pi/K\alpha} \exp\Big(K\alpha(b - \frac{1}{2} + \frac{(1-b)^2}{2a} + \frac{1}{2}\log a)\Big) \\ &\times \exp\Big(-\frac{1}{2}\alpha a \sum_k \omega_k^2 + j\alpha b \sum_k \omega_k\Big), \end{aligned} \tag{24}$$

where the resulting ω dependent integrand is a Gaussian function. Thus performing Gaussian integration and exploiting the symmetry in the K-dimensional space, we get

$$\begin{aligned} \mathcal{N}(\beta,\gamma) &= \lim_{K\to\infty} \frac{2^{K-K'}}{\pi^{K'}} \int_{-\infty}^{\infty} \frac{da\ db}{4\pi/K\alpha} \exp\Big(K\alpha(b - \frac{1}{2} + \frac{(1-b)^2}{2a} + \frac{1}{2}\log a)\Big) \\ &\times \exp\Big(K'\log\Big(\sqrt{\frac{2\pi}{\alpha a}} \int_{-\alpha}^{\infty} d\lambda \exp\big(-\frac{(\alpha b + \lambda)^2}{2\alpha a}\big)\Big)\Big). \end{aligned} \tag{25}$$

Using the rescaling $(\alpha b+\lambda)/\sqrt{\alpha a}\to\lambda$, the integral (25) becomes

$$\mathcal{N}(\beta,\gamma)=\lim_{K\to\infty}\int_{-\infty}^{\infty}\frac{da\ db}{4\pi/K\alpha}\exp\left(Kg(a,b,\gamma,\beta)\right),\tag{26}$$

where the function $g(a,b,\gamma,\beta)$ is defined by

$$g(a,b,\gamma,\beta)\triangleq\frac{1}{\beta}(b-\frac{1}{2}+\frac{(1-b)^2}{2a}+\frac{1}{2}\log a)+\gamma\log\left(2Q(t)\right)+(1-\gamma)\log(2),\tag{27}$$

with an auxiliary variable

$$t\triangleq\frac{\sqrt{\alpha}(b-1)}{\sqrt{a}}.\tag{28}$$

Again, for $K\to\infty$, the double integral in (26) can be evaluated by the saddle-point method. Hence, we find[2]

$$\mathcal{N}(\beta,\gamma)\propto\lim_{K\to\infty}\exp\left(Kg(a^*,b^*,\gamma,\beta)\right),\tag{29}$$

where a^* and b^* are found by the saddle-point conditions, which yield the following equations

$$\frac{\partial g(a,b,\beta)}{\partial a}=\beta^{-1}(\frac{(1-b)^2}{a}-1)+\gamma t\frac{Q'(t)}{Q(t)}=0,\tag{30}$$

$$\frac{\partial g(a,b,\beta)}{\partial b}=\beta^{-1}(1-\frac{1-b}{a})+\gamma\frac{1}{\sqrt{a\beta}}\frac{Q'(t)}{Q(t)}=0.\tag{31}$$

The operator Q' denotes a derivative of Q w.r.t. its argument. This set of saddle-point equations can be solved numerically to obtain its fixed-points a^*, b^* and t^*.

Finally, substituting (29) into (8) the asymptotic entropy, in nats, is now easily obtained

$$\begin{aligned}H(\beta,\gamma)&=g(a^*,b^*,\gamma,\beta)\\&=\frac{1}{\beta}(b^*-\frac{1}{2}+\frac{(1-b^*)^2}{2a^*}+\frac{1}{2}\log a^*)+\gamma\log\left(2Q(t^*)\right)+(1-\gamma)\log(2),\end{aligned}\tag{32}$$

which, along with equations (28),(30)-(31), concludes our proof and forms the desired theorem.

5 Results

Fig. 1 presents the optimum AME of PO-CDMA $\eta_{\text{opt}}(\beta)=\max_\gamma\eta(\beta,\gamma)$, drawn via an exhaustive search over all possible values of the active users fraction γ. Also drawn for comparison are the AMEs of the single-user matched filter (SUMF), decorrelator, and linear minimum mean square error (LMMSE) detectors [2].

Interestingly, $\eta_{\text{opt}}(\beta\lesssim 0.1)$ is practically equal to 1, which is the optimum AME of randomly spread CDMA [7], obtained for the optimal MUD. For large β loads the optimum AME of the examined scheme converges to 1/4, and not 0 as for the other sub-optimal detectors. The corresponding optimal fraction γ is found to decrease from 1 to 1/2 as we increase β.

[2]The exponent pre-factor in (29) is not required for computing the asymptotic entropy, and therefore it is omitted.

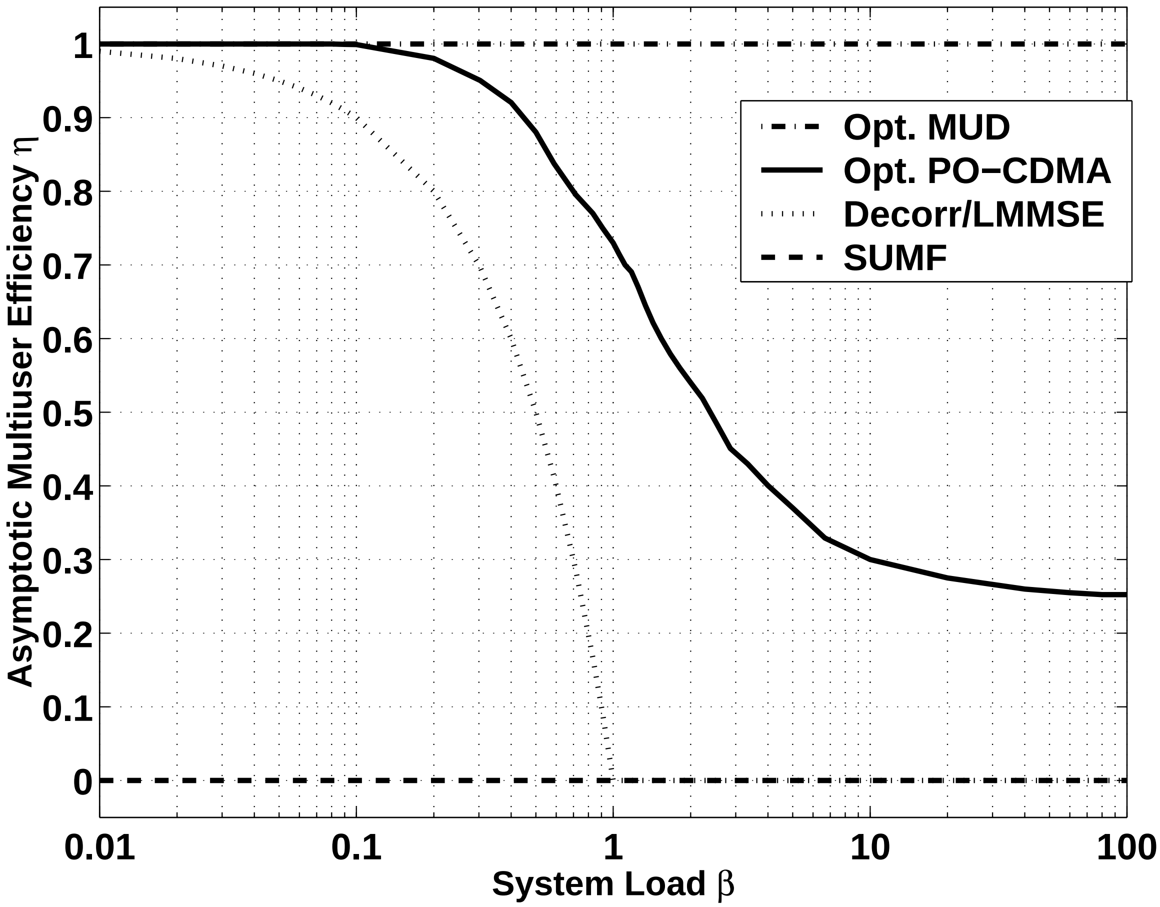

Figure 1: Optimum AME of PO-CDMA compared to optimal MUD and linear detectors.

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