Matrix Games with Nonuniform Payoff Distributions

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In this study we examine the relation between the probability distribution of the elements in the payoff matrix in 2-person zero sum games, and the statistical properties of those games. We generalize the results which were obtained for large random payoff matrices, to cases in which the probability distribution of the matrix elements depends on the column they belong to. We take advantage of methods from statistical mechanics to find analytical expressions for the value of the game and for the distribution of the strategy strengths. The analytical results are compared with simulations.

Recently, the statistical mechanics has crossed the borders of physices, and methods from statistical mechanics are being used to solve wide variety of problems from different fields including error correcting codes [1], cryptography [2] and game theory [3,4]. Game theory is the mathematical study of conflict situations which opens up a wide range of game-like problems in economics, sociology, psychology, and strategic studies to the precise interpretation of mathematical analysis. It was established in 1928 by J. von Neumann who went with O. Morgenstern to lay the foundations for virtually the whole of what is now called the mathematical theory of games [5]. One fundamental class of problems in game theory is 2-person zero sum games. These games describe situations in which two persons with completely opposite interests play against each other [6]. In such games, player X may choose between N strategies and player Y between M strategies. The game is defined by a payoff matrix C of size $N*M$ such that at each step, when strategy i of X and strategy j of Y are played, player X receives the payoff c_{ij} whereas the payoff of player Y is $-c_{ij}$. In order to maximize their payoffs, the players may choose the strategy which guarantees the highest payoff in the worst case. Following this scenario, player X chooses strategy i^* satisfying $\min_j c_{i^*j} = \max_i \min_j c_{ij}$. Equivalently player Y chooses strategy j^* determined by $\max_i c_{ij^*} = \min_j \max_i c_{ij^*}$. If both players play in this way, and $\max_i c_{ij^*} = \min_j c_{i^*j}$, a saddle point exists and neither has cause for regret once the opponent's choice is known.

However, not all matrix games have saddle points. For such games, premature disclosure of a choice of pure strategy would be to a player disadvantage. A way out of the difficulty consists of deciding not on a single pure strategy but on a choice between pure strategies regulated by a chance. Such a probability combination of the original pure strategies is called a mixed strategy. A mixed strategy of player X is a vector of probabilities $\mathbf{x} = (x_1, ..., x_N)$ and by the normalization condition is constrained to lie on the N-dimensional simplex. Similarly, $\mathbf{y} = (y_1, ..., y_N)$ is a mixed strategy of player Y. According to the minimax theorem by von Neumann, for any matrix C there exists a saddle point of mixed strategies $\mathbf{x}^*$ and $\mathbf{y}^*$ such that

$$\max_{\{\mathbf{x}\}} \min_{\{\mathbf{y}\}} \sum_{ij} x_i c_{ij} y_j = \sum_{ij} x_i^* c_{ij} y_j^* = \min_{\{\mathbf{y}\}} \max_{\{\mathbf{x}\}} \sum_{ij} x_i c_{ij} y_j \tag{1}$$

Given the optimal mixed strategies $\mathbf{x}^*$ and $\mathbf{y}^*$, the expected gain of both players $\nu_c = \sum_{ij} x_i^* c_{ij} y_j^*$ is called the value of the game.

In a recent study, J. Berg and A. Engel [3] have shown how methods from statistical mechanics can be used to investigate the statistical properties of optimal mixed strategies of large matrix games with random payoff matrices. They have found analytically the behavior of the value of the game and the fraction of played strategies as a function of the ratio $\alpha = M/N$. As expected, the value of the game decreases with α since as α increases, player Y who wish to minimize the payoff, becomes more powerful. Moreover, it has been shown that the number of played strategies of player X increases with α. In our study we use the same methods to find how the statistical properties of the matrix games are influenced by the probability distribution of the elements in the payoff matrix. The matrix elements are considered to be independent Gaussian distributed variables, while without the loss of generality, we take a fraction p of the matrix columns to have either A times the variance of the other columns, $A = << {c_{ij}}^2 >>$, or a shift A in the average. The average and the variance of the other columns are taken to be $1/N$ and zero respectively.

The fraction of mixed strategies obeying $\Sigma_i x_i c_{ij} \geq \nu \forall j$ is given by the partition function

$$Z(\nu) = \frac{\prod_{j=1}^{\alpha N} (\int_0^\infty dy_j) \delta(\sum_{j=1}^{\alpha N} y_j - N) \prod_{i=1}^{N} \Theta(\nu - \sum_j y_j c_i j)}{\prod_{j=1}^{\alpha N} (\int_0^\infty dy_j) \delta(\sum_{j=1}^{\alpha N} y_j - N)} \tag{2}$$

where $\Theta(x)$ is the Heaviside step function and the probabilities of playing a given strategy and the payoff have been rescaled so that $\Sigma_{j=1}^{N} y_j = N$. The meaning of $Z(\nu)$ is the relative fraction of mixed strategies which obey the constraint that $\sum_j y_j c_i j \le \nu \; \forall i$. Given $Z(\nu)$, the entropy $S(\nu) = 1/N \ln(Z(\nu))$ can be calculated using the replica-trick $lnZ = lim_{n\to 0}\frac{d}{dn}Z^n$. This identity when averaged over the payoffs allows to replace the hard task of calculating $<< \ln Z >>$ with the calculation of the averaged nth power of the partition function. To examine the influence of nonuniform variance on the features of the game we use the same procedure that has been used to obtain $Z(\nu)$ in reference [3], with $q_{ab} = 1/N \sum_{j=1}^{\alpha p N} y_j^a y_j^b + A^2/N \sum_{j=\alpha p N+1}^{\alpha N} y_j^a y_j^b$. The partition function for player Y yields

$$Z^n(\nu) = \prod_{a\ge b}\int_{-\infty}^{\infty}\frac{dq_{ab}d\hat{q}_{ab}}{2\pi/N}\prod_a\int_{-\infty}^{\infty}\frac{dE_a}{2\pi/N} \tag{3}$$

$$\exp(-iN\sum_{a\ge b}q_{ab}\hat{q}_{ab} - iN\sum_a E_a)$$

$$\prod_{a,i}\int_{-\infty}^{\infty}d{x_i}^a \exp(i\sum_{a\ge b,i}\hat{q}_{ab}{x_i}^a{x_i}^b + i\sum_{a,i}E_a{x_i}^a)$$

$$\prod_{a,j}\int_{\nu}^{\infty}d{\lambda_j}^a\int_{-\infty}^{\infty}\frac{d{y_j}^a}{2\pi}exp(-\frac{1}{2}\sum_{a,b,j}q_{ab}{y_j}^a{y_j}^b + i\sum_{a,j}{y_j}^a{\lambda_j}^a)$$

For large payoff matrices, $N\to\infty$, one can use the saddle point method and the replica symmetry ansatz:

$$q_{aa} = q_1 \quad iq_{aa} = -\frac{1}{2}\hat{q}_1 \quad iE_a = E \quad \forall a \tag{4}$$

$$q_{ab} = q_0 \quad iq_{ab} = -\frac{1}{2}\hat{q}_0 \quad \forall a > b$$

to calculate $Z^n(\nu)$. The analytical continuation $n \to 0$ gives an entropy:

$$S(\nu) = extr_{q_1,q_0,E,\hat{q}_1,\hat{q}_0}[\frac{1}{2}q_0\hat{q}_0 + \frac{1}{2}q_1\hat{q}_1 - E \tag{5}$$

$$+\int Dt\ln(1-H(\frac{\sqrt{q_0}t+\nu}{\sqrt{q_1-q_0}})) + \frac{\alpha}{2}\ln 2\pi - \frac{p\alpha}{2}\ln(\hat{q}_1+\hat{q}_0)$$

$$-\frac{(1-p)\alpha}{2}\ln(A^2(\hat{q}_1+\hat{q}_0)) + p\alpha\frac{\hat{q}_0+E^2}{\hat{q}_1+\hat{q}_0} + (1-p)\alpha\frac{A^2\hat{q}_0+E^2}{A^2(\hat{q}_1+\hat{q}_0)}$$

$$+p\alpha\int Ds\ln(H(-\frac{\sqrt{\hat{q}_0}s+E}{\sqrt{\hat{q}_1+\hat{q}_0}}))$$

$$+(1-p)\alpha\int Ds\ln(H(-\frac{A\sqrt{\hat{q}_0}s+E}{A\sqrt{\hat{q}_1+\hat{q}_0}})) - 1$$

where $G(s) = 1/\sqrt{2\pi}\exp(-s^2/2)$, $Ds = dsG(s)$ and $H(x) = \int_X^\infty Ds$. Similarly to the case of uniform variance [3], as $\nu \to nu_c$ $S(\nu)$ tends to $-\infty$ and $q_0 \to q_1$. We rescale the order parameters such that $\hat{q}_0$ becomes $v^2\hat{q}_0$ and $\hat{w}$ becomes $v\hat{w}$ where $v = q_1 - q_0$ and $\hat{w} = \hat{q}_1 + \hat{q}_0$. As $v \to 0$, the saddle point equations become:

$$\hat{w} - (1 - H(-\frac{\nu_c}{\sqrt{q_0}})) = 0 \tag{6}$$

$$-\hat{q}_0 - \sqrt{q_0}\nu_c G(-\frac{\nu_c}{\sqrt{q_0}}) + \hat{w}(q_0+\nu_c^2) = 0$$

$$\hat{w} - \alpha[pH(-\frac{E}{\sqrt{\hat{q}_0}}) + (1-p)H(-\frac{E}{A\sqrt{\hat{q}_0}}) = 0$$

$$E = \hat{w}q_0 - \hat{q}_0$$

$$\hat{w}(E-1) + \alpha\hat{q}_0[pG(-\frac{E}{\sqrt{\hat{q}_0}}) + \frac{(1-p)}{A}G(-\frac{E}{A\sqrt{\hat{q}_0}})] + \frac{(1-p)(1-A^2)}{A^2}\alpha E$$

where we keep terms up to first order in t. The influence of the variance on the value of the game can be easily observed by writing the saddle point equations for the maximizer. The three parameters which determine the behavior of ν_c are A, p and α. Setting p to zero it turns out that $\nu_c(A,p,\alpha) = A\nu_c(1,0,\alpha)$ indicating that for $\alpha > 1$ the value of the game decreases with A. For $p > 0$ and $\alpha > 1$, one distinguishes between two different behaviors of ν_c as a function of A. For $\alpha p > 1$, as A decreases ν_c increases until $\nu_c(A_c,p,\alpha) = \nu_c(1,0,\alpha p) > 0$ and then stays constant in the region $0 \le A \le A_c$. This behavior of ν_c is understandable since if the minimizer chooses only between the fraction p of strategies which have a variance 1 and ignores the other part of the matrix, he can guarantee a payoff of at least $-\nu_c(1,0,\alpha p)$. However if $\alpha p < 1$ then $\nu_c(1,0,\alpha p)$ is negative and hence never equals $\nu_c(A,p,\alpha)$. In this case the value of the game decreases to zero as $A \to 0$. For $\alpha < 1$, $\nu_c(A,p,\alpha)$ increases with A and approaches 0 as $A \to 0$ independent of αp. Figure 1 demonstrates the two possible behaviors of nu as a function of A.

The statistical properties of the optimal strategies are obtained by $\Theta(a)$, the relative fraction of pure strategies played with a probability greater than a. To calculate this quantity for the minimizer, $\Theta_{min}(a)$, one has to average $\frac{1}{\alpha N}\sum_{j=1}^{\alpha p N}\Theta(y_j^* - a)$ over the random matrices C. This calculation gives

$$\Theta_{min}(a) = pH(\frac{\hat{w}a-E}{\sqrt{\hat{q}_0}}) + (1-p)H(\frac{\hat{w}a-E}{\sqrt{A\hat{q}_0}}), \tag{7}$$

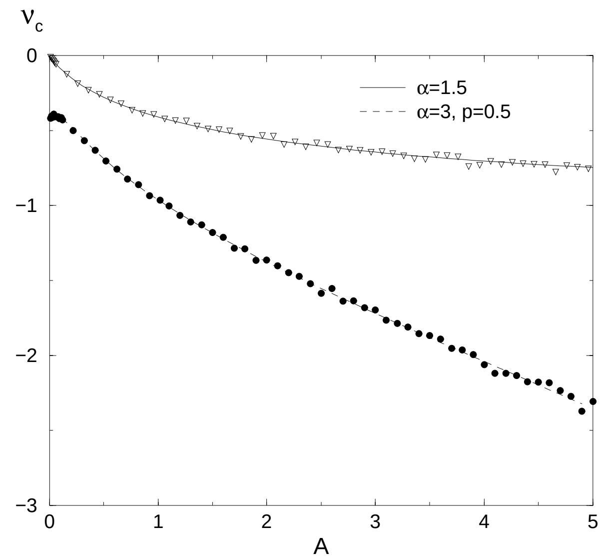

FIG. 1. The two types of behavior of the value of the game vs. A. In both cases $p = 0.5$. The dashed-line describes the numerical solution of Eq. 6 for $\alpha = 3$ ($\alpha p > 1$) whereas the numerical solution for $\alpha = 1.5$ is presented by the solid-line. Simulation results for $N = 200$ averaged over 100 samples with $\alpha = 3$ (circles) and $\alpha = 1.5$ (triangles) are in agreement with the analytical solution.

indicating that the columns with the higher variance are played with a higher probability than the other strategies as shown in Fig. 2. Calculating the same quantity for the maximizer it turns our that $\Theta_{max}(0)/\Theta_{min}(0) = \alpha$ indicating that the same number of strategies is played by each one of the players. Note that the maximal number of played strategies is obtained for the most balanced case in which $\alpha = 1$.

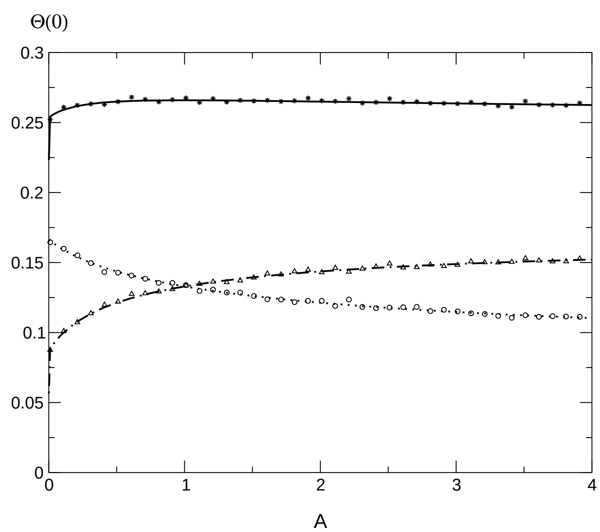

FIG. 2. $\Theta_{min}(0)$ vs. A for $\alpha = 1.5$ and $p = 0.5$ as calculated from Eq. (7) (solid curve) compared with simulation results (stars) with $N = 200$ averaged over 100 samples. The relative fraction of played strategies with variance 1, $\Theta^1_{min}(0)$, decreases with A as described by the dotted-line (analytical solution) and the circles (simulation results). The long-dashed line (analytical solution) and the triangles (simulations) stand for the fraction of played strategies with variance A, $\Theta^A_{min}(0)$, which increases with A.

The value of the game and the statistical properties of the optimal strategies can be easily derived for the general case in which the variance of the columns is determined by a function $P(A)$ which represents the fraction of the columns characterized by a variance A. For such cases the only modification in the saddle point equations (6) is in equations 3 and 5 which become:

$$\hat{w} - \sum_A P(A) H(-\frac{E}{A\sqrt{\hat{q}_0}})$$

and

$$\hat{w}(E-1) + \alpha\sqrt{\hat{q}_0}\sum_A P(A) G(-\frac{E}{A\sqrt{\hat{q}_0}}) + \sum_A \frac{(1-p)(1-A^2)}{A^2}\alpha E H(-\frac{}{A}$$

respectively. $\Theta_{min}(a)$ (Eq. 7) changes in the same manner.

We now turn to study the case of nonuniform average of the matrix columns. We consider the matrix elements to be Gaussianaly distributed with variance $1/N$, where a fraction p of the columns have an average 0 and the other columns have an average A. For such cases the saddle point equations for the maximizer become:

$$\hat{w} - H(-\frac{E}{\sqrt{\hat{q}_0}}) = 0 \quad (8)$$

$$-\hat{q}_0 + \alpha\sqrt{q_0}(p\nu_c G(-\frac{\nu_c}{\sqrt{q_0}}) + (1-p)(\nu_c - \tilde{A}) G(-\frac{\nu_c - \tilde{A}}{\sqrt{q_0}})) + p(q_0 + \nu_c^2) H($$

$$\hat{w} - \alpha[pH(-\frac{\nu_c}{\sqrt{q_0}}) + (1-p)H(-\frac{\nu_c - \tilde{A}}{\sqrt{q_0}}) = 0$$

$$E = \hat{w} q_0 - \hat{q}_0$$

$$q_0 - \frac{\sqrt{q_0}E}{\hat{w}} G(-\frac{E}{\sqrt{\hat{q}_0}}) - \frac{\hat{q}_0 + E^2}{\hat{w}} = 0,$$

where $\tilde{A} = AN$ and the relative fraction of strategies played with a probability greater than a is $H(\frac{\hat{w}a - E}{\sqrt{\hat{q}_0}})$.

Obviously, for $p = 0$ a shift A in the average of all matrix elements results in a shift AN in the value of the game (the factor N is due to the normalization condition). For $p = 1$, ν_c equals to its value in the uniform case of $A = 0$. Solving numerically Eq. 8 it turns that ν_c increases with A (see Fig 3).

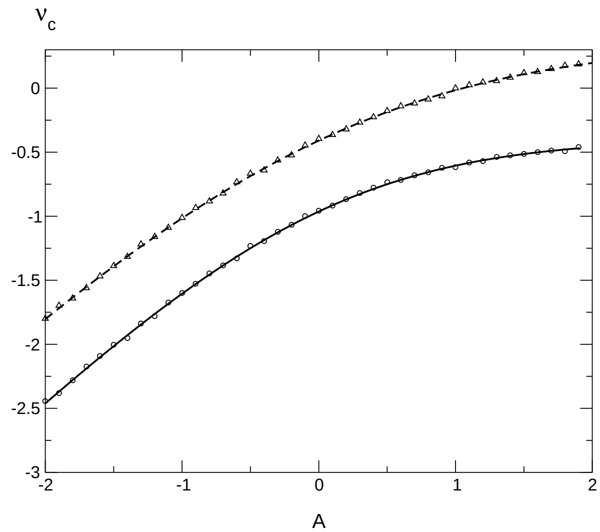

FIG. 3. The behavior of ν_c as a function of A as obtained from the numerical solution of Eq. 8 for $p = 0.5$ with $\alpha = 3$ (solid-line) and $\alpha = 1.5$ (dashed-line). Simulation results (symbols) for $N = 200$ averaged over 100 samples are in good agreement with the analytical prediction.

Moreover as $A \to \infty$ $\nu_c \to \nu_c(0, 1, \alpha p)$ and all the played strategies are from the columns with average 0, whereas as $A \to -\infty$, $\nu_c \to A + \nu_c(0, 1, \alpha(1-p))$ and all the played strategies are from the columns with average A. To explain the behavior of ν_c at the two tails of A, one should keep in mind that if only strategies with average 0 are played then ν_c equals to its value for $p = 1$ and $\alpha = \alpha p$. On the other hand, in case that the minimizer chooses to use only strategies with average A, then the expected payoff is $A + \nu_c(0, 1, \alpha(1-p))$. Obviously, as $A \to \infty$, it is preferable for the minimizer to choose all the strategies from the $1-p$ part of the matrix which yields $\nu_c = A + \nu_c(0, 1, \alpha(1-p))$. As A increases, $\nu_c(A, p, \alpha)$ increases but as was pointed out before, since the minimizer can guarantee a payoff of $\nu_c \to \nu_c(0, 1, \alpha p)$ by choosing all his strategies from the p part of the matrix, ν_c cannot exceed this value. Fig. 4 exhibits the fraction of played strategies of the minimizer for $\alpha = 3$ and $p = 0.5$. As expected, the fraction of played strategies with 0 average tends to 0 as A decreases and approaches $\Theta_{min}(0)$ in the large A regime.

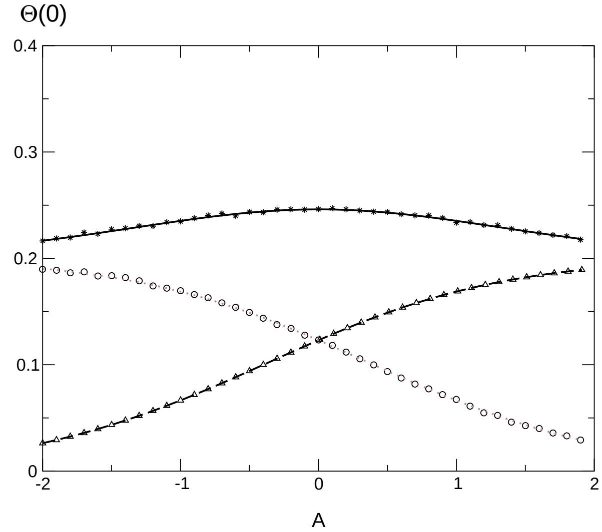

FIG. 4. $\Theta_{min}(0)$ vs. A for $\alpha = 3$ and $p = 0.5$. The solid curve stands for the analytical result and the stars represent simulations with $N = 200$ averaged over 100 samples. The relative fraction of played strategies with average 0, $\Theta^0_{min}(0)$, increases with A as described by the dotted-line (analytical solution) and the circles (simulation results). The long-dashed line (analytical solution) and the triangles (simulations) stand for the fraction of played strategies with average A, $\Theta^A_{min}(0)$, which increases with A (see explanation in the text).

To conclude, in this work we study zero sum games with large matrices characterized by nonuniform average and variance. We find analytical expressions for the value of the game and for the statistical properties of the optimal strategies and compare these results to the uniform case which has been studied in [3]. Further research is necessary to study other modifications of matrix games such as cooperative games and multi-player games [6].

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