Cascading Parity-Check Error-Correcting Codes

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A method for improving the performance of sparse-matrix based parity check codes is proposed, based on insight gained from methods of statistical physics. The advantages of the new approach are demonstrated on an existing encoding/decoding paradigm suggested by Sourlas. We also discuss the application of the same method to more advanced codes of a similar type.

Error-correcting codes are commonly used in most means of information transmission. The coding efficiency, measured in the percentage of informative transmitted bits, plays a crucial role in determining the speed of communication channels and the effective storage space on hard-disks.

The question of channel capacity of noisy communication channels was addressed by Shannon [1] in his pioneering work from 1948. Rigorous bounds have been derived for the maximal transmission rate for which codes, capable of achieving arbitrarily small error probability, can be found.

In a typical scenario, a message comprising N binary bits is transmitted through a noisy communication channel; the received string differs from the transmitted one due to noise (for instance, background radiation, defects in magnetic materials, thermal noise etc) the result of which is the flipping of some bits. Here, we denote the flipping rate of a bit in a binary symmetric channel (i.e., from 0 to 1 or from 1 to 0) by $f \in [0:1]$; other types of noise may also be considered, which may be more realistic in some scenarios. Error-correcting codes have been devised for retrieving the original message at the receiving end.

The receiver can correct the flipped bits (or some fraction of them) in a retrieved message of length N only if the source transmits $M(f) > N$ bits; the ratio between the original number of bits and those of the transmitted message $R \equiv N/M$ is termed the code-rate. Shannon [1] obtained the optimal trade-off between the following three quantities in the thermodynamic limit: the maximal code rate R_c (termed channel capacity), the flip rate f (due to noise) and the coded bit error probability p_b, given explicitly by

$$R_c = (1 - H_2(f))/(1 - H_2(p_b)), \tag{1}$$

where $H_2(x) = x \ \log_2(x) + (1-x) \ \log_2(1-x)$.

Shannon's theory indicates the existence of optimal codes but does not provide a way of obtaining them. Many algorithms were devised to overcome this practical problem (for a review see [2]), however, the performance of most algorithms is below Shannon's bound.

One error-correcting code which recently became popular is the Gallager code [3–6]. In this method, the transmitted message comprises the original message itself and additional bits used for error correction. Each one of the additional bits is generated by summing up randomly selected message bits; the parity of the sum constitutes the transmitted code-word bit. The choice of the message-vector elements used for generating single code-word bits is carried out according to a predetermined random set-up and may be represented by a product of a randomly generated sparse matrix and the message-vector in a manner explained below. It was shown that by using Gallager's method it is possible to get closer to Shannon's bound of the maximal channel capacity for specific choices of the encoding/decoding matrices [4–7].

In this paper we will introduce a method for constructing the encoding/decoding matrix employed in general Gallager-type codes that enables one to improve the code's performance significantly. In this method the matrix comprises specifically constructed sparse matrices, designed to gradually build up the overlap between the original and the decoded message. We will demonstrate to potential of the method by examining the model suggested by Sourlas [8], representing a particular case of Gallager's code. The performance of this simple model is generally inferior to that of other advanced Gallager codes. However, we prefer to present the main ideas behind our method via the code of Sourlas [8] due to its straightforward relation to Ising spin models and the transparent interpretation of the construction. More general constructions have been employed recently for improving the performance of more complicated Gallager type codes [7] bringing their performance close to saturating shannon's bound.

In a general scenario, a message $\boldsymbol{s}$ is encoded to a code-word $\boldsymbol{t}$, which is then transmitted through a noisy channel. The code-word is corrupted during transmission by noise, represented by the vector $\boldsymbol{n}$, and the received code-word $\boldsymbol{r}$ is decoded by the receiver for retrieving the original message.

Sourlas's approach is based on mapping the coding problem onto that of an Ising spin system. The original presentation [8] made use of a binary (± 1) message vector representation; here, for brevity and consistency with notation commonly used for Gallager's method, we will mostly use the Boolean (0,1) formulation of the problem unless stated otherwise. In this approach one constructs code-word bits by taking the sum of randomly selected K Boolean message bits (mod 2)

$$\mathbf{t} = A \ \boldsymbol{s} \ (\text{mod } 2)$$

were the matrix A contains $K(\ll N)$ unit elements per row and $C(=KM/N)$ per column, setting all other elements to zero.

Using a simple linear transformation, $\hat{s}_i = (2s_i - 1)$, the Boolean bits $s_i \in (0,1)$ can be mapped onto binary ones $\hat{s}_i \in (-1,1)$. The physical system then consists of N Ising spins with K spin interactions and a fixed connectivity C, i.e., each spin participates in C interactions. The corresponding Hamiltonian has the following form:

$$H = -\sum_{\langle i_1, i_2 \ldots i_K \rangle} J_{i_1,i_2,\ldots,i_K} \; S_{i_1} S_{i_2} \ldots S_{i_K} \;, \qquad (2)$$

where $\{S_i\}$ are the binary dynamical variables used in the decoding process, which can take the values (± 1). The noisy free channel interaction tensor $J^0_{i_1,i_2,\ldots,i_K} = \hat{s}_{i_1}\hat{s}_{i_2}\ldots\hat{s}_{i_K}$ where $\hat{s}$ is the binary representation of the originally Boolean message vector $\boldsymbol{s}$; the choice of indices $i_1, i_2, ..., i_K$ is predetermined between the sender and the receiver, reflecting the non-zero row elements of the matrix A. Due to corruption during transmission $J_{i_1,i_2,\ldots,i_K} = \hat{s}_{i_1}\hat{s}_{i_2}\ldots\hat{s}_{i_K}$ with probability $1-f$ and $-\hat{s}_{i_1}\hat{s}_{i_2}\ldots\hat{s}_{i_K}$ with probability f. Under a gauge transformation this model is mapped onto a highly diluted Ising spin system with ferromagnetic bias (we assume $f \leq 0$). The magnetization $m = 1/N \; \sum_{i=1}^{N} s_i \hat{s}_i$ is related to the number of correct bits $(1+m)/2$.

Finding the ground state of the Hamiltonian, in terms of the variables $\mathbf{S}$, corresponds to the Bayes optimal estimation of the original message bits and thus to decoding the received message [9]; it can be carried out using various techniques, including energy minimization (simple Monte-Carlo at some temperature, say zero) or belief propagation (e.g. [4,10]). The properties of Sourlas's method have been investigated for the fully connected [8] and diluted cases [11] with fixed connectivity.

The main drawback of the method is the need to compromise between superior capabilities and poor decoding performance, and sub-optimal capabilities (in terms of the achievable code rate) and successful decoding. For example, the choice of $K=2$ in Sourlas's approach, i.e., having only 2 Multi-Spin Interactions (MSI), corresponds to an energy landscape dominated by a very large basin of attraction; this will lead to a successful convergence from almost *any* small positive initial overlap between the dynamical variables and the message, and will result in a large end-overlap m (and consequently successful decoding). However, this overlap is much smaller than 1, the *perfect* decoding required in most cases, even for flip rates way below Shannon's bound. On the other hand, choosing higher K (and consequentially higher C) values may result in very high end-magnetization and successful decoding, but also in a corresponding dramatic decease in the basin of attraction. The improved end-magnetization can be easily understood as the increased connectivity reduces the probability of a negative local field asymptotically; the reduced basin of attraction clearly results from the vanishing contribution of the product of K spins far from the ground state. Consequently, one may expect a decoding failure unless the starting point is chosen very close to the original message; such information is clearly unavailable in practical scenarios. One should emphasize that the basin of attraction shrinks dramatically, for instance, for $K=6$ the initial magnetization required for successful convergence is higher than $0.98 - 0.99$.

Our method builds on insight gained from the study of physical systems with symmetric and asymmetric [12] multi-spin interactions and previous studies of Sourlas's code via methods of statistical physics [11]. It is based on the gradual introduction of higher connectivity sparse matrices, exploiting the excellent convergence properties of codes based on low K values with the high performance of high-K codes. For example, one may carry out the first stage of the decoding process using $K=2$ and then, once the overlap between the decoded word and the original message is within the relevant basin of attraction, one invokes the $K=3$ connections (that were already used in generating the code-word), resulting in a much higher overlap in comparison to the case in which only $K=2$ connections are used. The process can clearly be generalized to include a longer sequence of transitions and to different K-values, such as to improve the overall performance. One should point out that from a physical point of view this is equivalent to changing the Hamiltonian (2) as a function of time.

It has been shown that the method does not have to be implemented in a dynamical manner as the one described above as long as the encoding/decoding structured matrix is constructed appropriately. The dynamical implementation is slightly superior close to the critical flip rate. It also enables one to obtain, at zero temperatue, results which are typically obtained only at finite temperatures.

An optimal construction of the encoding/decoding matrix is clearly the key point to a successful algorithm. Although there is no clear recipe for constructing the matrix in general, one can provide a few guidelines that are helpful for improving the performance.

Originally, the method relies on invoking the next level connections once the current state of the system (and the resulting overlaps with the input message) is within its basin of attraction. The latter can be estimated numerically either by an exhaustive search or approximated analytically, by considering contributions to a single node and averaging over the input probabilities. This approximation assumes magnetization m per contributing node, where $\langle S_i(t) \rangle = m$ for all i neglecting correlations among the different sites (spins). The average $\langle \cdots \rangle$ represents an average over possible spin distributions and weights, where the prior on the weights is taken as $P(J) = f \; \delta(J+1) + (1-f) \; \delta(J-1)$ and $P(m) = (1+m)/2 \; \delta(m-1) + (1-m)/2 \; \delta(m+1)$. The basin of attraction at zero temperature is then defined as the minimal magnetization such that $\langle \mathrm{sign}(h) \rangle \geq m$, where h is the induced field. The end magnetization obtained is defined as the m value for which the equality holds. To demonstrate the agreement between results

obtained numerically and analytically and how they reflect the essence of the cascaded decoding approach, we examine analytically the case of $R=1/3$ and $f=0.14$: (a) Having only 2 MSI and $C=6$ $(=K/R)$ interactions one obtains a value of $m=0.932$ for the end magnetization in comparison to $m\approx 0.94$ obtained empirically; no significant limitation on the basin of attraction has been observed, i.e, convergence is expected from any initial overlap. (b) For $(C=)$ 4 interactions with $(K=)$ 2 MSI and $C=4$ interactions with 4 MSI (this provides in total the same message length as before $1/R=3=4/2+4/4$) one obtains an end magnetization of $m=0.97$; the value obtained empirically is $m\approx 0.98$. The basin of attraction requires an initial overlap $m\geq 0.6-0.62$. (c) For only 4 interactions with 2 MSI (i.e., as in (b) but omitting all 4 MSI components) one obtains a final magnetization of $m=0.64$ with no restrictions on the basin of attraction.

This forms the basis for the cascading error-correcting method: Starting from any (positive) initial overlap between the message and the dynamical variables, and employing the configuration of (c), one obtains an end magnetization which is well within the basin of attraction of the complete Hamiltonian system (b). Following with the dynamics of the complete system results in an end magnetization of $m\approx 0.98$, well above the end magnetization of (a), although the two systems have the same code-rate and initial basin of attraction.

The optimal combination of MSI, for which the highest end magnetization is obtained, depends on many parameters including the code rate R, the noise level and the message bias. Finding the optimal connectivities ratio of two different MSI values can be carried out by plotting the end magnetization of a partial system with only low MSI connections (such as (c) with $C=4$ and $K=2$) against the minimal magnetization required for convergence (basin of attraction) in the complete system (as in (b)). Figure 1 shows the two curves as a function of the fraction $0\leq\rho\leq 1$ of $K=4$ interactions (and with $(1-\rho)$ $K=2$ interactions). The experiment has been carried out for the case of $R=1/3$, $f=0.14$ and $N=10^4$ and the results were averaged over 10 trials. From figure 1 it is clear that any choice of $\rho\leq 0.42$ will lead the partial configuration to a higher end magnetization than required for the complete system to converge. As systems with higher connectivity will result in higher end-magnetization, one should aim at choosing the highest ρ value for which the partial system's end-magnetization is higher than the basin of attraction of the complete system, i.e., the intersection of the two graphs.

In constructing the matrix one may have to use a non-integer effective number of connections per spin C (i.e., the number of non-zero elements in a column of the matrix A) due to its relations to other system parameters; denoting the number of message bits with K_k MSI as N_k than $\sum_k N_k = 1/R$ and $C=\sum_k K_k N_k/N$ where N is the number of message elements. We have found it useful to keep the distribution of non-zero elements per column as homogeneous as possible to provide equal corrective contribution to all bits. In addition, it would be helpful to avoid having small loops in the connectivity matrix, i.e., small groups of sites connected cyclically, as these contribute to recurrent dynamics which suppresses corrective input from the rest of the system.

For converging to the correct final state it is useful to initialize the system with some positive overlap between the dynamical variables and the original message as in most of these systems, both solutions, with $m=\pm 1$, are equally attractive. This may be achieved either by transmitting a small fraction of the message itself simultaneously, which is the less favorable solution, or by adding some odd-MSI (i.e., an odd K value, e.g., $K=3$) to the mainly even K value used initially (e.g., $K=2$); this assists in breaking the symmetry from any initial dynamical variables setting with practically no effect on the basin of attraction. Odd connectivity per spin also alleviates the problem of zero local field; it is characterized by finite zero temperature entropy and hence improves convergence and the performance in general.

As decoding is carried out iteratively, it is important to define a halting criterion for obtaining the decoded message. Here we carry out a simple energy minimization, and the algorithm comes to a halt when all spins are aligned to their local fields (except spins of zero local fields). The local field of each spin h_i is calculated in turn by summing over all other spin states

$$h_i = \sum_{\langle i_2 \ldots i_K \rangle} J_{i,i_2,\ldots,i_K} \; S_{i_2} \ldots S_{i_K} \; .$$

The binary value of the individual spin is then obtained by aligning it with the value calculated for h_i, i.e., $S_i = +1$ if $h_i > 0$ and $S_i = -1$ otherwise. This dynamical process is repeated until the system stabilizes or until some halting criterion is obeyed.

To show the excellent performance of the new method we compared the end-overlap of four different systems of rate $R=1/7$ (i.e., code-word length of $M=7N$, corresponding to a critical flip rate $f_c=0.282$ due to Shannon, see Eq.(1)), starting from similar initial conditions and the same transmission flip rate $f=0.25$. Experiments have been carried out for different system sizes, $N=5000-20000$, for getting a feel for the dependence of the performance on the system's size. The dynamics employed for the energy minimization is based on sequentially updating the spins although similar results were obtained for parallel dynamics. The results summarized in table I show a significant improvement in the final overlap due to the cascaded encoding/decoding scheme. The low MSI part of the code word serves to bring the decoded vector to a sufficiently high overlap with the original message, so that it lies within the basin of attraction of the combined message; the final convergence to a very high overlap is facilitated by the high MSI.

These results, can be improved upon by having a reliable prior knowledge of the noise level. No such knowledge was assumed in any of the above mentioned exper-

iments. One should also point out that in all the experiments, we observed convergence after a few tens of iterations at most and the complexity of the algorithm used is of $O(N)$. The physical interpretation to the success of this method benefits from viewing the system as a graph whereby the different nodes (message sites) are sparsely connected by unit weights (elements of the multidimensional tensor $\boldsymbol{J}$). By increasing the number of MSI, say from $K=2$ to $K=4$, one increases the graph connectivity and the number of inputs which contribute to determining the state of each specific spin. On the other hand, as the number of MSI increases, their 'quality' deteriorates; the average local field is determined by taking the product of $K-1$ terms representing the magnetization of all spins connected to the relevant weight, which decays rapidly with the increase in MSI (e.g., a magnetization of 0.6 in the case of $K=2$ will reduce to $0.6^3=0.216$ in the case of $K=4$). Furthermore, increasing the number of MSI creates exponentially many local minima and energy levels which are highly degenerate (for instance, one local field obtained in a system with K MSI has a degeneracy of 2^{K-1}), making the probability of successful convergence vanishingly small. This interplay is at the center of our approach and guides the choice of the optimal model parameters.

To conclude, we have shown that through a successive change in MSI and connectivity one can boost the performance of matrix based error-correcting codes. We showed that this is feasible for the special case of Gallager-type codes presented by Sourlas [8], although the method itself is applicable to all codes of this type and may be easily adapted to fit most of the existing variations as will be shown elsewhere [7]. There are a few extensions that we would like to point out:
1) Although our examples concentrated on unbiased messages, the process can clearly be easily generalized to biased messages. It may also be generalized to include non-symmetric connections and continuous or multilevel message units (instead of binary).
2) It is plausible that many sets of parameters have similar performance in the thermodynamic limit, however, their finite size behavior above and below saturation is of great interest from a practical point of view. Finding architectures that are superior in their finite size behavior, as well as finding methods to suppress the finite size effects, would clearly be of great practical significance.

The cascading decoding method and the extensions mentioned above open a wide range of possibilities for a highly efficient encoding/decoding mechanism of significant practical value.

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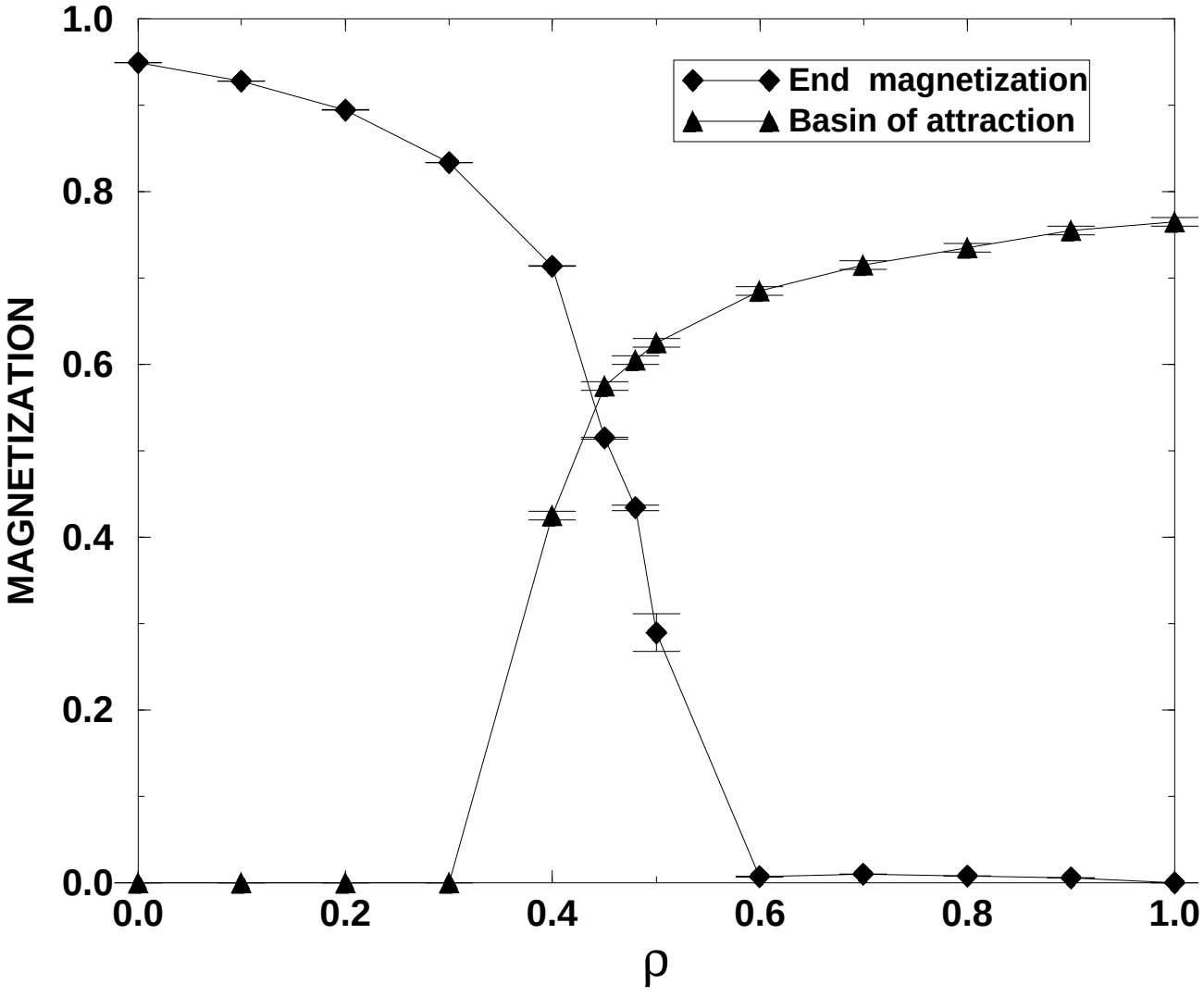
Message bits (N_k)	MSI (K_k)	Final magnetization
$7N$	2	0.940
$5N$	2	0.975
$2N$	5	
$5N$	2	0.987
$2N$	7	
$5N$	2	0.993
$2N$	9	

TABLE I. Final overlap for various combinations of MSI in the case of $R=1/7$ and transmission flip rate of $f=0.25$, starting from similar initial conditions for the different configurations.

FIG. 1. The end magnetization of a partial system (diamonds) with only low MSI connections ($C=4$ and $K=2$ and $M=3N$) against the minimal magnetization required for convergence (basin of attraction) in the complete system (triangles). The two curves are plotted as a function of the fraction $0\leq\rho\leq 1$ of $K=4$ connection (and $(1-\rho)$ connections of $K=2$).


End magnetization
Basin of attraction
MAGNETIZATION
ρ
0.0
0.2
0.4
0.6
0.8
1.0