Undecidability Principle and the Uncertainty Principle Even for Classical Systems

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It is shown that for any physical system there is an infinite number of measurements which are related to one of the undecidable problems. The undecidability of the physical systems indicates that there is an infinite number of incomputable correlation functions for each physical system. These results show that there is an inherent and irreducible limitation on the knowledge of the nature of physical systems. This uncertainty principle is caused by the undecidability principle.

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A basic question is whether the description of any physical property of a system is limited even in the classical limit. This is a general question which does not relate to the model describing the system. Fundamentally, any question related to a physical property is relevant, but its importance is subjective (depending on the answer, applications, etc.). If a question cannot be answered, then there is an uncertainty with respect to the discussed physical property. In the last few years, several studies have focused on a few *specific* incomputable properties of a classical continuum fluid[1] and cellular automata.[2]

This paper will generalize previous works by showing that for *any* physical system, there is an irreducible limitation on our knowledge of the properties of the system. The source of this limitation is the undecidability principle, indicating that there is an infinite number of incomputable correlation functions for each physical system. Unlike Heisenberg's uncertainty principle,[3,4] this principle is not based on any assumption and, furthermore, exists even in the classical limit. However, at this stage we would like to stress that this new source of uncertainty cannot be an alternative explanation for quantum phenomena, such as interference which is well understood by the wave nature of particles.

Let us now examine the new source for the limitation on the knowledge of even the properties of classical systems.

One of the well-known undecidable problems is the "domino-snake" problem.[5] The inputs of the problem are a finite set T of oriented square-tile types (dominos) and two locations A (x_1,y_1) and B (x_2,y_2) on the infinite two-dimensional square lattice. We assume that the Euclidean plane is divided into unit squares with centers at $\{(x_i,y_i)\}$. The problem asks whether it is possible to connect A and B by a domino snake consisting of tiles from T, with every two adjacent tiles (on the 1D snake) having monochromatic touching edges (see Fig. 1). The path between A and B might twist and turn erratically, reaching unboundedly distant points before converging on B.

The decidability of the domino-snake problem depends on the portion of the plane available for laying down the connecting tiles. If the portion of the plane is finite, there are only finite possible snakes that can be positioned in this finite area. Therefore, for a finite portion of the plane the problem is trivially decidable. If snakes are allowed to go anywhere, the problem is decidable and can be solved in a time that is polynomial in the distance between A and B and in the number of different tiles.[6] Nevertheless, if the allowed area for the snakes is only a half of the plane, the problem becomes undecidable. The analog of this problem to higher dimension is also undecidable.[5]

The problem on the half plane seems from the first impression to be more bounded; however, the fact is quite different. The domino-snake problem[7] shows us that unboundedness does not imply undecidability.

The undecidability of the domino-snake problem means that there is no way we can construct any algorithm to be run on any computer (regardless of the amount of time and memory space) to decide whether an arbitrary finite set of tiles can connect A and B.

Let us now try to clarify the connection between such an undecidable problem and physical problem.

The natural related problem is an anisotropic 1D Potts Hamiltonian which is given by

$$H=-\sum_{i=1}^{N}\sum_{r,s}^{q}J^{rs}\delta_{\sigma_i,r}\delta_{\sigma_{i+1},s}\ , \tag{1}$$

where N is the size of the system, σ_i is a Potts variable which could be in q different states $1,2,\ldots,q$, and the matrix J^{rs} is in the general case an asymmetric matrix. Let us concentrate now on matrices whose elements can take only two different values, 0 or 1. In such a general

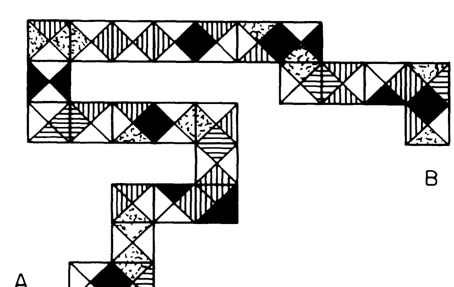

FIG. 1. A domino snake from A to B.

case and in the thermodynamic limit, it is obvious that the zero-temperature entropy per spin is finite. To see this, suppose that σ_i is in a state r. Then, σ_{i+1} could be in a few possible different states even at zero temperature. Furthermore, systems with finite zero-temperature entropy are related to undecidable problems as we have pointed out elsewhere.[8] More precisely, one can find a mapping between the snake problem with a given set of tile types and the 1D anisotropic Potts systems in the following way: (a) The number of Potts states, $k(p)$, is equal to 4 times the number of tiles; that is, $k=1, 2, \ldots, T$ and $p=1,2,3,4$. Each set of four states, $k(p)$, where $p=1,2,3,4$, represents tile k in four different positions (up, down, left, and right) with respect to the previous tile. (b) If a tile T_l can be connected to tile T_k from the side up, for instance, then $J^{k(p),l(1)}=1$, $p=1,2,3,4$. A similar result holds for sides down, left, and right, with $l(1)$ replaced by $l(2)$, $l(3)$, and $l(4)$, respectively. (c) For each pair of adjacent Potts spins (σ_i,σ_{i+1}) one can assign the following quantities: $\hat{\mathbf{z}}$ if the connection between the pair of tiles T_iT_j is forbidden, $+\hat{\mathbf{x}}$ $(-\hat{\mathbf{x}})$ if the tile T_j can be on the right (left) of tile T_i, and $+\hat{\mathbf{y}}$ $(-\hat{\mathbf{y}})$ if the tile T_j can be up (down) from the tile T_i. Hence, to each segment of the 1D chain, $[i,i+k]$, one can assign a number. This number is the sum of the k numbers which are assigned to the last k pairs of spins $(\sigma_i,\sigma_{i+1}), \ldots, (\sigma_{i+k-1},\sigma_{i+k})$. In the case that the segment is a part of one of the degenerate ground states of the system then the number has the following form: $l\hat{\mathbf{x}}+m\hat{\mathbf{y}}$ [which we denote later as (l,m)]. The question in the Potts system equivalent to the domino-snake problem is, according to this reasoning, whether one of the degenerate ground states in the thermodynamic limit of (1) contains a segment $[A,B]$ which obeys the following constraints: (a) For each site k, which belongs to the segment, one can relate a number (l_k,m_k). Spins with the same (l,m) should be in the same main Potts state $[k(p), p=1,2,3,4]$. (b) For all the spins in the segment, $\{l_k\}$ should be greater than or equal to zero, where $l_A=0$. This constraint is equivalent to the constraint that the allowed area for the snakes is only the half plane (the upper plane, for instance). (c) For the last spin σ_B, $l_B=x_2-x_1$ and $m_B=y_2-y_1$. The question of whether one can find such a segment in the Hamiltonian (1) for any given number q and a matrix J^{rs} is obviously also an undecidable problem. It is also clear that this question is related to a specific measurement of correlations among the spins which obeys the constraints of the domino-snake problem. However, the undecidability of the problem indicates that there are correlation functions among the spins for which there is no way to decide whether they are possible or impossible in one of the degenerate ground states of the chain system. This limitation is the uncertainty principle even of classical systems. We will return to discuss this property more carefully later on.

At this point one might object that the undecidability principle is not general. One might be tempted to argue that it is related only to the following specific systems: (a) systems which are described by anisotropic Potts Hamiltonians, (b) systems with a finite zero-temperature entropy, (c) systems in 1D, (d) systems at zero temperature, (e) static properties of spin systems, (f) only spin systems, and (g) only systems with discrete elements and discrete time. Nevertheless, in the following we will show that the undecidability principle applies even to the systems which are excluded in (a)-(g). In general, as we will explain, the undecidability principle applies to any physical system.

Let us first assume a uniform 1D Ising system, where the Hamiltonian is given by

$$H=-J\sum_{i=1} s_is_{i+1}\,, \tag{2}$$

where $s_i=\pm 1$ and the length of the chain is infinite. Let us explain now how this system is related to the domino-snake problem with a finite set of input tiles. To each tile from the set of T different tiles, one can assign a state from a block of at least

$$\lceil \log_2 T \rceil \tag{3}$$

Ising spins. Here again, for each pair of blocks of spins one can assign a number from $\pm\hat{\mathbf{x}}$, $\pm\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ in correspondence to the domino problem. Hence, to each segment of blocks one can assign a number of the form (l,m,n). The directions $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ are related, as we mentioned above, to the directions of the connections between pairs of tiles. The undecidability is related to the following question. Does the phase space of the system (2) contain a state such that a segment of it obeys the following constraints: (a) For all the blocks in the segment the related number is of the form $(l,m,0)$; (b) all blocks with the same $(l,m,0)$ should be in the same state; (c) for all the blocks, $\{l_k\}$ should be greater than zero; (d) the number $(l,m,0)$ of the last block of the segment is such that $l=x_2-x_1$ and $m=y_2-y_1$? The question of whether the phase space contains such a point is an undecidable problem.

We would like to stress that for each different set of input tiles for the domino problem there corresponds a different anisotropic Potts Hamiltonian. The matrix J^{rs} is changed according to the input type of tiles. Therefore, the undecidability is related only to a part of the possible 1D anisotropic Potts Hamiltonians, because there are systems in which it is trivial to find the answer for any given A and B. Nevertheless, the example (2) indicates that the undecidability applies to all the discussed systems with any type of interactions. For any given type of tiles one can find the appropriate blocks in any given system which lead to an undecidable problem (regardless of the type of the interactions). Furthermore, the number of different undecidable problems one can relate to each system is *infinite*. To see this let us first stress that the size of the blocks, Eq. (3), is bounded only from below by $\lceil \log T \rceil$. Hence, one can define a larger block than $\lceil \log T \rceil$ and relate the T tiles to a subset of the possible states of the block. The undecidable problem is related now to a segment of blocks, each one of which is in one of the T states, and they are connected according to the constraints we explained above. Each different undecidable problem is related either to a different subset of T states or to a different size scale of the block. Hence, the number of incomputable correlation functions for each system is *infinite*.

It is obvious that one can extend this picture to an Ising system in a higher number of dimensions. For instance, in two dimensions one can relate to each tile a configuration of a square tile containing at least $\lceil \log_2 T \rceil$ Ising spins. In such a case the equivalent question is whether the phase space of the 2D Ising system contains a sequence of blocks which obey the constraints of the domino-snake problem.

At a finite temperature there is a related question. What is the probability that the system in equilibrium (or in nonequilibrium) is in a state which is related to the undecidable problem? The undecidability of the problem indicates again that there is no way to answer this question.

We would like to stress now that the ground state of systems which are related to the undecidable problems are not necessarily degenerate. For instance, in the Hamiltonian (1) one can add to each interaction J_i^{rs} some small random number ϵ_i such that $\epsilon_a < |\epsilon_i| \ll J^{rs}$. In this case the ground state is unique. The undecidability problem in this case is whether the phase space contains a sequence of spins as before with the additional constraint that the contribution to the energy function of each pair of spins is greater than $J - \epsilon_a$.

One can ask similar questions on the dynamical evolution of such stochastic spin systems. For instance, does the dynamical evolution of a spin i (for instance) in the Hamiltonian (1) contain a segment with the same constraints as for the static question? This question is also obviously an undecidable problem. It is important to note that the same undecidable question is related even to the dynamical evolution of a single particle, like one Ising spin in the presence of a noise.

All these systems and the appropriate undecidable problems are related to some spatial or temporal correlation functions. The undecidability of these problems indicates that even in *classical systems* the optimal knowledge of the correlation functions is limited. There are many (more precisely, infinitely many) different correlation functions of which we cannot decide whether they can exist in the system at zero or finite temperatures. These results are very general. They apply to systems in any number of dimensions, to both static and dynamical properties of such systems, and to ordered or disordered spin systems. The undecidability applies even to systems with any range of interactions. The easiest way to see this is to fix some arbitrary order among the spins and the system then becomes a 1D system for our purpose. Hence, the undecidability applies to any discrete-spin systems. Furthermore, one can apply the undecidability even to systems with continuous spins and continuous time. To illustrate, quantize the time and the space of each spin. For instance, in the case of Heisenberg spins, one can decide arbitrarily that if during the next second, for instance, s_{ix} is positive, then $\sigma_i = 1$ and otherwise $\sigma_i = -1$. It is now obvious how to relate the problem in the new spin variables σ to the domino-snake problem. However, the configuration in the σ variables is related to corresponding correlations in the original s spin variables.

Basically, the undecidability or the uncertainty principle applies to any physical system and not only to spin systems. For instance, consider a gas molecule in a container. If during the next second the molecule spends more time in the left half of the container, we indicate it as 1, otherwise we indicate it as 0. We can again relate a block of a few seconds to the type of input tiles, and hence to the undecidability of the domino-snake problem.

To illustrate this idea let us assume an experiment where the motion of a Brownian particle is examined by an observer under a microscope which can be moved in space.[9] The observer is then asked the following question. Let us divide the view screen of the microscope into squares with some arbitrary length. Furthermore, a finite set of rules are given to the observer which are related to the constraints of the domino-snake problem, as we discussed above. The observer is then asked whether it is possible to find a Brownian trail which obeys the given constraints under any length scale. The right answer of a clever observer should state clearly, "I will try to answer the question." However, it is possible that regardless of the given time, the budget, the experimental tools, and my cleverness, I will never be able to find the right answer.

Undecidability limits our knowledge on the spatial or temporal correlation functions even of classical systems. Hence, there is an intrinsic limit on the knowledge of the static and the dynamical properties of classical systems. This limitation is the *uncertainty principle* for classical systems. The uncertainty is intrinsically embedded in the systems. It does not matter how accurate our measurements are. There are quantities that cannot be measured because of the uncertainty principle which is caused by the undecidability principle.

It is also obvious that this source of uncertainty appears even in the quantum level. Similar questions which lead to the undecidability of the gas molecule in a container can be asked of any quantum system.

It is important to restress that the mathematical tools and the experiments which describe the *statistical properties* of nature could become more and more accurate. For instance, to calculate the average magnetization of a 1D Ising system it is completely unnecessary to know the exact properties of all configurations of the phase space. However, it is impossible to know all properties of the configurations space. Furthermore, Heisenberg's uncertainty is based on the assumption that the outcome of a measurement has a statistical nature. By contrast with this uncertainty, the discussed uncertainty principle is more fundamental and unconditioned. It states that there are possible outcomes for which there is no way to decide whether the probabilities are zero or positive.

The discussed analogy between the Ising system and undecidability also indicates that there are infinitely many points for which there is no way to decide whether they exist in any line segment. Hence, the accuracy of any measurements is limited.[8] A measurement is meaningful only when two things are compared. A comparison means that the two objects occupy the same points in space. However, there is no way to answer this question, because these undecidable points may distinguish between the objects.

Many undecidable problems, other than the domino-snake problem, also correspond to some incomputable physical measurements. One of these problems is the tiling problem. The input of the problem is some finite set T of oriented cards. A card is a unit square divided into four by the two diagonals and each quarter colored by some color. The problem asks whether the infinite plane can be covered using only cards from T, such that the colors on any two touching edges are the same. This problem is also an undecidable problem,[10] and the physical analogy of this problem is also quite clear.[8] We would also like to emphasize that the uncertainty principle is not necessarily related to such yes or no questions, as appear in the above-mentioned problems. An example will clarify this point: The problem asks for the *size* of the largest area that can be tiled legally by any set of tiles that involves no more than N colors, but cannot be extended spatially in any way. The output of this undecidable problem should be a number and not a yes or no answer. It is clear, as we discussed above, that this problem is related to some correlation-length measurements and to the structure of the domain walls in the equivalent physical system.

It is important to note that the undecidable problems are known to be grouped in infinite hierarchies. Each level contains problems that are worse than those residing on lower levels. Each undecidable problem which belongs to a specific level can be decided with the aid of an imaginary oracle (which solves the problem) for any one of the other problems belonging to the same level. For instance, the domino problem and the tiling problem belong to the same level. Nevertheless, the tiling problem with the additional constraint that a specific tile will appear infinite times in a legal tiling is known to be in a higher level of undecidable problems.[11] Hence, this irreducible limitation of physical systems is not unique like the Heisenberg uncertainty principle but grouped in infinite hierarchies. As mathematically undecidable problems, so are physical measurements grouped in infinite hierarchies. Perhaps future research will reveal whether there is any analogous connection between the quantum measurements and their level of undecidability.

Finally, this work raises many basic questions which were formally considered to be trivially answered. A more detailed discussion will be given elsewhere,[8] but let us mention briefly a few of these questions: What is the definition of classical physical reality? Can one find an analog of Bell's theorem[12] for the classical undecidable principle? What is the analog and the consequences, if any, of a zero-point motion for classical systems? What is the measure of the undecidable physical quantities with respect to the full space of physical reality?

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