PHYSICAL REVIEW A VOLUME 38, NUMBER 11 DECEMBER 1, 1988

Asymmetric neural networks with multispin interactions

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(Received 29 August 1988)

A dilute nonsymmetric ferromagnetic neural network with multispin interactions and a version of the Hopfield model with multispin interactions are solved in the limit where the average number of inputs per spin is finite. It is found that in such asymmetric systems the critical exponents of local quantities can take a few different values, instead of just one.

Spin-glass models for associative memory have been of increasing interest in the last few years. As first proposed by Little[1] and Hopfield,[2] these models were based on an Ising Hamiltonian and hence can be treated by equilibrium statistical mechanics.[3] Two assumptions are crucial to allow for an exact solution of the equilibrium properties of the model. Each neuron is assumed to be connected to an infinite number of other neurons, and the synaptic connections are assumed to be symmetric. Here a symmetry (asymmetry) indicates that the influence of neuron i on neuron j is equal (not equal) to the influence of neuron j on neuron i. Besides the relevance to biological systems, asymmetric systems have important applications in computer science and in statistical dynamical systems which are far from equilibrium.

In this paper, we first give a solution for the dynamics of a highly dilute nonsymmetric ferromagnetic network and a nonsymmetric Hopfield model with k-spin interactions.

The model consists of a system of N Ising spins $S_i=\pm 1$. The strength of the interactions depend on p stored patterns and are given by

$$J_{i_1,i_2,\ldots,i_k}=C_{i_1,i_2,\ldots,i_k}\sum_{\mu=1}^{p}\xi_{i_1}^{\mu}\xi_{i_2}^{\mu}\cdots\xi_{i_k}^{\mu}\,. \tag{1}$$

The symbol $J_{i_1,i_2,\ldots,i_k}$ stands for an asymmetric interaction from $i_1\rightarrow i_2\rightarrow\cdots\rightarrow i_k$ with the strength $J_{i_1,i_2,\ldots,i_k}$. The p patterns $\{\xi_i^{\mu}\}$ are certain configurations of the network which were fixed by the learning process. They are assumed to be random, with equal probability for $\xi_i^{\mu}=\pm 1$. The system will have the capacity of storage, if indeed the emergent dynamical configurations are correlated with the patterns. The $C_{i_1,i_2,\ldots,i_k}$ are random independent parameters which represent the dilution and the asymmetry in a system with k-spin interactions. For each group of k spins, a $C_{i_1,i_2,\ldots,i_k}$ is chosen at random according to the distribution $\rho(C_{i_1,i_2,\ldots,i_k})$, which is given by

$$\rho(C_{i_1,i_2,\ldots,i_k})=\frac{c}{N^{k-1}}\delta(C_{i_1,i_2,\ldots,i_k}-1)+\left[1-\frac{c}{N^{k-1}}\right]\delta(C_{i_1,i_2,\ldots,i_k})\,. \tag{2}$$

It is important to note that the interactions $J_{i_1,i_2,\ldots,i_k}$ are not symmetric. For each group of k spins, there are $k!$ independent C's.

For the models (1) and (2), the following dynamics can be considered.[4] Parallel dynamics for which, at time t, all spins are updated simultaneously according to the following rules: On each site i_1, the induced local field $h_{i_1}(t)$ is computed via

$$h_{i_1}(t)=\sum_{i_2,\ldots,i_k}J_{i_1,i_2,\ldots,i_k}S_{i_2}S_{i_3}\cdots S_{i_k}\,. \tag{3}$$

Notice that each interaction through k spins, contributes to the local field of only one spin. The spins are updated according to the following rule

$$S_i(t+\Delta t)=\begin{cases}+1, & \text{with probability } \{1+\exp[-2h_{i_1}(t)/T]\}^{-1},\\ -1, & \text{with probability } \{1+\exp[2h_{i_1}(t)/T]\}^{-1},\end{cases} \tag{4}$$

where for parallel dynamics, the natural time scale is $\Delta t=1$. We also consider random sequential dynamics for which, at time t, one randomly chooses at a spin i among N spins and updates this spin according to Eqs. (3)-(4). In this case, one should scale the time with the size of the system and $\Delta t=1/N$.

In this paper, we obtain exact results for the dynamical properties of the model in the thermodynamic limit, where c according to Eq. (2) is the average number of inputs per spin. The capacity and the quality of the retrieval as a function of the concentration of the bonds in the system are obtained. The result that in these systems the critical exponents of local quantities can take a few different values is also discussed.

We will consider the evolution of a configuration $\{S_i(t)\}$ which has a macroscopic overlap

$$m(t)=\frac{1}{N}\sum_{i=1}^{N}\xi_i^1 S_i(t) \tag{5}$$

with one stored pattern and microscopic overlaps with the other $p-1$ random patterns. For clarity we will present the details of the calculations only in the ferromagnetic case ($p=1$), and we concentrate mostly on the zero-temperature limit.

Following Ref. 4 the dynamical evolution of $m(t)$ is given for parallel dynamics by

$$m(t+1)=f(m(t)), \tag{6}$$

and for random sequential updating by

$$\frac{dm(t)}{dt}=f(m(t))-m(t), \tag{7}$$

where $f(m)$ is given by

$$f(m)=\sum_{n=0}^{\infty}\frac{e^{-c}c^n}{n!}g_n(m). \tag{8}$$

For the case of odd $2k+1$ multispin interactions

$$g_n(m)=n!\left\langle\left\langle\prod_{l=0}^{k}\left[\binom{2k}{l}\left(\frac{1+m}{2}\right)^{2k-l}\left(\frac{1-m}{2}\right)^{l}\right]^{\rho_l}\frac{1}{\rho_l!}\delta\left(\sum\rho_l-n\right)\tanh\left(\beta\sum\rho_l(-1)^l\right)\right\rangle\right\rangle_{\{\rho_l\}}, \tag{9}$$

where the average is over ρ's which are non-negative integers. There is a similar equation for the case of even number of multispin interactions which, for simplicity, is not presented here.

The derivation of Eqs. (8) and (9) is very similar to what has been done in Ref. 4. Briefly it is derived as follows. First, as long as c is small,

$$c\ll\log N, \tag{10}$$

the magnetization in the quenched and annealed models are both given by Eq. (9).[5-7] This is due to the fact that the loops in the systems are of $O(\log N)$ and the inputs of almost all sites are not correlated by any finite time. Therefore, under the restrictions of Eq. (10) the calculation can be carried out for the annealed model. The term

$$e^{-c}c^k/k! \tag{11}$$

in Eq. (7) is the density of spins having k nonzero inputs. This result indicates that unlike symmetric systems, in asymmetric systems there are two sources which are responsible for creating noise in the system. The first source is the temperature of the system, and the second source is the finite fraction of spins, e^{-c}, which are free of inputs. The term in the square brackets of Eq. (9) is proportional to the probability that there are ρ_l inputs, where each one of them has $2k-l$ spins equal to 1 and l spins equal to -1. The δ function stands for the constraint that the total number of inputs are equal to n. The last term is the local field for a fixed realization of the ρ's. The contribution to the local field from each multispin interaction containing even number of spins equal to -1 is positive, and this with odd number of spins equal to -1 is negative.

In the zero-temperature limit, when one averages over the randomness in the system, one can find that the average fixed magnetization for a system with k-spin interactions (k is odd or even number) is given by

$$m=ce^{-c}\int_{(1-m^k)/2}^{(1+m^k)/2}dt\frac{\mathrm{I}_1(2ct_1)}{2t_1}+\mathrm{I}_0(2ct_1) \tag{12}$$

where I_ν are modified Bessel functions and $t_1\equiv\sqrt{t(t-1)}$. This result holds for both parallel and random sequential dynamics. Numerical solution of Eq. (12) shows that as a function of c the system undergoes at c^* a strong first-order phase transition. This result is in contrast to the case $k=2$, where the system undergoes a second-order phase transition.[8] For example, for $k=3$, 5, 10, $c^*\simeq3.66$, 4.86, 6.23 where the magnetization at the transition is $m^*=0.894$, 0.97, 0.993, respectively. The increasing function of c^* as a function of k, is due to the fact that one coupling induces, on the average, a nonzero local field *only* if all spins which belong to it have a nonzero magnetization. In the large k limit, c^* is given implicitly by $k\simeq e^{c^*}/2c^*(1+c^*/2)$ and $m^*\to1$.

As a function of the temperature one can also verify that these systems undergo a first-order phase transition, where in the case $k=2$ the system undergoes a second-order phase transition.[4]

The results for the ferromagnetic case at zero temperature with $k=3$ and 5 were checked in simulations on systems containing from 1000 to 2000 spins. The results from the simulations for $k=3$ and 5 were $c^*=3.67\pm0.03$, 4.9 ± 0.1, $m^*=0.93\pm0.02$, 0.98 ± 0.05, respectively. These results are very close to the theoretical results in the thermodynamic limit, and indicate that the finite size effects are small.

In a similar calculation one can show that in the Hopfield case with p embedded patterns [see Eq. (1)], the average fixed magnetization is given by

$$m=\int_{-\infty}^{\infty}\frac{dx}{i\pi x}\exp\{c[-1+\cos(x)+im^{k-1}\sin(x)][\cos(x)]^{p-1}\}, \tag{13}$$

and in the limit c and $p\to\infty$ and $\alpha\equiv p/c$, ($c/N\to0$) one can find

$$m=\mathrm{erf}(m^{k-1}/\sqrt{2\alpha}). \tag{14}$$

Numerical solutions of Eq. (14) show that the system undergoes at α^*, a strong first-order phase transition. At the transition, α^* is a decreasing function of the number of multispin interactions and m^* is an increasing function of the number of multispin interactions. For example, for $k=3$, 4, 8, $\alpha^*\simeq0.25$, 0.20, 0.14, and $m^*\simeq0.85$, 0.93, 0.98, respectively. In the case $k=2$, the system undergoes a second-order phase transition at $\alpha^*=2/\pi$.[4] In the large k limit one can find $\alpha^* \to 0$, but with a perfect retrieval.

The solution of asymmetric systems with multispin interactions give us the ability to examine the nature of the transition of complex asymmetric systems. Assume for simplicity, a system consists of two parts. The first part P_1 contains N_1 spins, and the second part P_2 contains N_2 spins. The induced local field on a spin $i \in P_1$ is

$$h_i = J_1 \sum_{j \in P_1} C_{ij}^1 S_j , \tag{15}$$

and the induced local field on a spin $i \in P_2$ is

$$h_i = J_2 \sum_{j,k,l \in P_2} C_{ijkl}^2 S_j S_k S_l + J_3 \sum_{j,k,l \in P_1} C_{ijkl}^3 S_j S_k S_l , \tag{16}$$

where the C's are random numbers which could take the values 0 or 1. The average of C_{ij}^1 and C_{ijkl}^q, $q=1,2$, are of $O(N^{-1})$ and $O(N^{-3})$, respectively [see Eq. (2)]. Under these restrictions, the inputs to each spin are not correlated.

This system consists of two layers, P_1 and P_2. The contributions to the local field in the first layer consist of two-spin interactions only among spins in the first layer. The contributions to the local field in the second layer come from two sources. The first one is four-spin interactions consisting of spins from the second layer. The second source is four-spin interactions consist of three spins from the first layer and the last spin from the second layer.

The simultaneous equations which are determined by the magnetizations m_1 and m_2 in the first and the second layer, respectively, can be derived in a similar way to Eqs. (6)–(9). One can find that there is a range of interaction strengths and probabilities distribution of the C's under which

$$m_1 \propto (T_c - T)^{1/2}; \quad m_2 \propto (T_c - T)^{3/2} , \tag{17}$$

where T_c is the transition temperature of the first layer. This result indicates that there are two different critical exponents β in the system. The exponent β for the first layer is $\frac{1}{2}$ and for the second layer is $\frac{3}{2}$.

This model can be extended to include a uniform magnetic field which is coupled to each spin in the system. In this case, the critical exponents γ_l and δ_l in the l's layer are defined by

$$\chi_l \equiv \frac{\partial m_l}{\partial h} \propto (T - T_c)^{\gamma_l}; \quad m_l(T_c) \propto h^{1/\delta_l} , \tag{18}$$

in the limit $h \to 0$. One can verify that in this case $\gamma_1 = 1$ and $\delta_1 = 3$ in the first layer, and $\gamma_2 = 0$ and $\delta_2 = 1$ in the second layer, which obey the scaling relations.

One can generalize this picture to infinite range-interacting systems with more layers, more types of multispin interactions, and for systems where there are *feedbacks* among the layers. In all these cases and for some range of interactions, one can find different values of a critical exponent for different classes of a *finite fraction* of spins in the system. This result indicates that a critical exponent which describes the nature of the phase transition is given in these cases by a *set* of exponents, and not by a unique value.

The different values of the critical exponent for different classes of a finite fraction of spins are also common to finite-dimensional asymmetric systems with a strong anisotropy. For simplicity, assume three symmetric or asymmetric separated layers where only the first two layers undergo a second-order phase transition at a finite temperature. It is obvious that if the third layer receives only inputs (does not sent outputs) from the first two layers, then its exponent β_3 could be any sum of two exponents from β_1, β_2, $1-\alpha_1$, $1-\alpha_2$. A critical exponent consisting only of β_l's, comes from spins which receive inputs from uncorrelated spins. Inputs from correlated spins lead to a critical exponent which depends on α_l. More precisely, the exponent α_l is a local quantity which is defined by $\langle S_x^l S_{x+n}^l \rangle \propto |T - T_c|^{1-\alpha}$, where the symbol $\langle \cdots \rangle$ stands for thermal average, and n is a finite number. These results exist even in the presence of only ferromagnetic interactions.

The number of necessary critical exponents of local quantities to describe the nature of the phase transition as a function of the asymmetry of the system is still unclear. Furthermore, the existence of such a behavior in finite-dimensional asymmetric systems without any special structure of anisotropy is also an important open question.

Another interesting question is the sensitiveness of spins in the system to the finite fraction of spins which are free of inputs. The importance of this question is well understood, if one looks at such a system as a feedforward neural network consisting of input units, output units, and "hidden" units.[8] The answer to this question depends very much on the initial conditions of the system. For simplicity, we concentrate here only on the case where the initial condition is chosen to be the case where all spins are equal to 1, and in the zero-temperature limit. In this case one can show that the fraction of spins, $m_1(t)$, that are equal to one, with probability one, in all first t steps per spin, is given by

$$m_1(t) = \sum_{n=0}^{\infty} \frac{e^{-c} c^n}{n!} \sum_{l > [n/2]} \binom{n}{l} [m_1(t-1)^{k-1}]^l [1 - m_1(t-1)^{k-1}]^{n-l} . \tag{19}$$

The derivation of Eq. (19) is based on the fact that a spin is equal to one with probability one if and only if more than one half of its inputs were equal to one with probability one in the last time step. The term $m_1(t-1)^{k-1}$ in Eq. (19) represents the probability that all spins which belong to one input of k-spin interaction are equal to 1 with probability one. In the limit $t \to \infty$ and $k=3$, for example, one can find a nonzero solution for m_1 only for $c > c^* \simeq 4.64$. The value of m_1 at c^* is ~ 0.96. This result indicates that there is a critical value c^*, above which the system is essentially insensitive to the noise coming from spins which are free of inputs.

One can immediately generalize this picture to more complicated assumptions for the spins which are free of inputs. Generalization to the case where the magnetization of each spin depends on the number of inputs and their connection strengths is also straightforward.[9]

It is important to stress that the capacity (number of retrieval bits) per bond in *randomly* diluted asymmetric systems with multispin interactions is of $O(1)$. Nevertheless, in the representation of the Pott's neural network with k states as a network of symmetric highly diluted multispin interactions, the capacity per bond was found to be proportional to $\log_2 k$.[10] This result indicates that the capacity per an asymmetric bond with multispin interaction might be also greater than $O(1)$ for some special dilutions.

In this paper, the dynamics of randomly diluted asymmetric networks with multispin interactions are solved exactly. We find the capacity and the quality of the retrieval as a function of the dilution. The critical exponents of local quantities were found to be able to take a few different values, instead of just one.

In summary, we would like to stress that networks which are not diluted at *random* may exhibit different behaviors. For instance, in the presence of correlated inputs, it is possible that the average magnetization of each input is zero, but the average magnetization of the spin which receives the inputs is nonzero. This behavior and some others may play an important role in biological systems.

I would like to thank specially Professor D. S. Fisher for numerous stimulating discussions and his critical reading of the manuscript. I would also like to thank Jonathan Yedidia for helpful conversations. This work was supported by the Weizmann Foundation and also in part by the National Science Foundation Grant No. DMR-8518163.

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