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LETTER TO THE EDITOR

The equivalence between discrete-spin Hamiltonians and Ising Hamiltonians with multi-spin interactions

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Abstract. An argument is presented to explain the fact that the same mean-field results apply to several spin glass models, such as the 'random-energy' model and the large-p Potts glass model. The dilute simplest spin glass model is solved, yielding the same mean-field results. The argument is based on the fact that the symmetry of the discrete-spin system can be determined by the discrete-spin type or by the form of the interactions. The mapping between the models can be extended to all forms of discrete-spin systems.

Magnetic systems with random competing interactions often condense at low temperature into a spin glass (SG) phase. The mean-field (MF) theory based on the Edwards–Anderson model [1] reveals that the transition of the low-temperture phase is a continuous transition, similar to an ordinary second-order phase transition. However, the SG phase is very unusual. It consists of many degenerate partially overlapping pure states characterised by an order parameter $q(x)$. The inverse of the order parameter, $x(q)$, determines the probability that two pure states have an overlap [2, 3] less than or equal to q.

Recently, there has been a growing tendency to apply concepts of SG theory to various infinite-range discrete SG models [4–9] which are a generalisation of the SK model [5]. One form of generalisation consists of a spin with more than two components. Another form of generalisation is the replacement of the random pair interactions by multi-spin interactions. Several of these generalised SK models [4–9] have a similar MF solution.

In this Letter, we show why models that differ in the spin representation and in the form of the interactions among the spins nevertheless exhibit a similar behaviour in their MF solution. The argument is based on the fact that each discrete-spin Hamiltonian can be mapped into an equivalent Ising Hamiltonian with multi-spin interactions. In this mapping each discrete spin of p discrete states is represented by a block of $\log_2 p$ Ising spins.

We present now examples of systems that have a similar MF solution. The system of N p-state Potts variables $\sigma(i) = 1, 2, \ldots, p$ is described by the Hamiltonian [4, 5]

$$H = \frac{-p}{2} \sum_{i \neq j}^{N} J_{ij} \delta_{\sigma(i), \sigma(j)}. \tag{1}$$

The random pair interactions, J_{ij}, are of infinite range and are quenched gaussian-distributed variables with mean $[J_{ij}] = J_0/N$ and variance $[J_{ij}^2] = J^2/N$. Related infinite-range models are the random chiral model introduced in [6]

$$H = -\tfrac{1}{2}\sum_{i\neq j}^{N}\sum_{r=0}^{p-1} J_{ij}^{(r)} \exp\left(\frac{2\pi \mathrm{i}\, r}{p}(\sigma(i) - \sigma(j))\right) + \mathrm{cc} \tag{2}$$

(with each $J_{ij}^{(r)}$ being an independent random variable) and the random anisotropic Potts model [4]

$$H = -\tfrac{1}{2}\sum_{i\neq j}^{N}\sum_{r,s=0}^{p-1} J_{ij}^{r,s}(\delta_{\sigma(i),r} - 1/p)(\delta_{\sigma(j),s} - 1/p) \tag{3}$$

(where each $J_{ij}^{r,s} = J_{ji}^{s,r}$ is an independent random variable). An example of a system where the random pair interactions are replaced by multi-spin interactions, with a similar MF solution, is the model of random p-spin interactions [7–9]. The Hamiltonian is

$$H_p = -\sum_{1\leqslant i_1<i_2\ldots<i_p}^{N} J_{i_1\ldots i_p} s_{i_1}\ldots s_{i_p} \tag{4}$$

where the spins s_i are Ising spins and the interactions $J_{i_1\ldots i_p}$, for any group of p spins, are quenched gaussian-distributed variables. The model of large-p multi-spin interactions is known as the simplest SG model [7, 8]. This model is identical [7] to the Derrida 'random-energy' model, defined as a system of 2^N independent random-energy levels distributed according to the probability distribution law

$$P(E) = (N\pi J^2)^{-1/2} \exp(-E^2/NJ^2). \tag{5}$$

Application of the Parisi replica theory shows that although all these models, equations (1)–(5), are very different, they exhibit a very similar low-temperature phase. Unlike in the SK model, the order function $q(x)$ is a discontinuous function. In particular, for a certain range of temperatures in the low-temperature phase, $q(x)$ is a single-step function

$$q(x) = \begin{cases} 0 & x<\bar{x} \\ q & x>\bar{x} \end{cases} \tag{6}$$

where

$$1 - \bar{x} \propto T_c - T. \tag{7}$$

This implies that the phase space splits into an infinite number of distinct SG (or Potts glass (PG)) phases which do not overlap.

In the cases of three-state or four-state infinite-range Potts systems [4] and two multi-spin intractions [9], the transition from the paramagnetic phase to the SG (or PG) phase is continuous, $q(1) = 0$ at T_c. For random systems with a higher order of multi-spin interactions [7–9] or Potts states [4], the transition from the paramagnetic phase to the SG (or PG) phase is discontinuous, $q(1) \neq 0$ at T_c. The 'single-valley' order parameter jumps discontinuously from zero at T_c, implying a discontinuous freezing of the local degrees of freedom in each of the valleys, which would be reflected by a discontinuity in the AC response to an external source [8].

We demonstrate now the similarity between the PG phase of the large-p random Potts systems and the 'random-energy' model with p^N states. These p^N independent random-energy levels are distributed according to the following probability distribution law:

$$P(E) = [\pi N(p-1)J^2]^{-1/2} \exp[-E^2/(p-1)NJ^2]. \tag{8}$$

For each sample of p^N energy levels one can define $n(E)$—the number of energy levels lying in the interval $(E, E + \mathrm{d}E)$. This number fluctuates from one sample to another. However, the average value $\langle n(E)\rangle$ over all choices of the energies is easily obtained from (8) to be

$$\langle n(E)\rangle = [\pi(p-1)NJ^2]^{-1/2} \exp[-E/(p-1)NJ^2 + N\ln p]. \tag{9}$$

Using the same argument as in [7], one can show that the free energy is given by

$$F/N = \begin{cases} -T\ln p - J^2(p-1)/4T & T > T_c \\ -J[(p-1)\ln p]^{1/2} & T < T_c \end{cases} \tag{10}$$

where

$$T_c = (J/2)[(p-1)/\ln p]^{1/2}. \tag{11}$$

Similarly, the energy of the system for $T < T_c$ is given by

$$|E_0|/N = J[(p-1)\ln p]^{1/2}. \tag{12}$$

Solving this model in the Parisi representation yields that $q(x)$ is the step function given by (6) and (7) with $q = 1$ and $x = T/T_c$, for all $T < T_c$. These results and (10)–(12) are exactly the same as the solution of the infinite-range randomly interacting large-p Potts models. Thus, this model of p^N independent random-energy levels is equivalent to the simplest SG model consisting of $N\log_2 p$ Ising spins. the models are not *completely* similar. For example, calculation of the probability distribution $P_{12}(E_1, E_2)$, the probability that two given configurations of spins have energies E_1 and E_2 respectively, gives for the random chiral model

$$P_{12}(E_1, E_2) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha\,\mathrm{d}B \exp[\mathrm{i}\alpha E_1 + \mathrm{i}\beta E_2 - NJ^2(p-1)(\alpha^2+\beta^2)/4 + J^4(p-1)\alpha^2\beta^2/4]. \tag{13}$$

This result shows that for the infinite-range Potts model

$$P_{12}(E_1, E_2) \neq P(E_1)P(E_2) \tag{14}$$

for all values of p. (Still, calculation of $P(E)$ for this model gives again the same result as (5).) Therefore, unlike the 'random-energy' model and the simplest SG model, in the Potts models there are correlations among the energy levels, even in the limit $p \to \infty$.

The similarity found for the MF solutions when $p \to \infty$ is found also for the finite p-states Potts model [4] and for finite p-spin interactions [9]. For these models, there are common ranges of temperatures for which $q(x)$ is the single-step function (6) and (7).

We now explain why a family of randomly interacting models exhibits similar behaviour in the MF solution and why the mapping between the systems is not limited only to random systems with an infinite range of interactions.

The replacement of each Potts spin by a block of $\log_2 p$ Ising spin is, in general, a necessary condition for the mapping between discrete-spin models. This is in agreement with the equivalency of the models discussed above. (For the cases for which $p \neq 2^R$, $R = 2, 3, \ldots$, one must add constraints or fields in order to avoid the forbidden states.)

In order to show more explicitly the similarity between the Potts model and the multi-spin interaction models, one should replace each Potts spin, $\sigma_i = 1, 2, \ldots, p$, by a block of $\log_2 p$ Ising spins $\{s_i^\mu\}_{\mu=1}^{\log_2 p}$. Then, using the identity

$$\delta_{s_i s_j} \equiv (1 + s_i s_j)/2 \tag{15}$$

one finds

$$-\tfrac{1}{2}\sum_{i \neq j}^{N} J_{ij}\delta_{\sigma(i),\sigma(j)} = -\tfrac{1}{2}\sum_{i \neq j}^{N} J_{ij} \prod_{\mu=1}^{\log_2 p} (1 + s_i^\mu s_j^\mu)/2. \tag{16}$$

In the block Ising Hamiltonian, there are multi-spin interactions (of order $2, 4, \ldots, 2\log_2 p$) between two blocks, but there are no multi-spin interactions connecting spins that belong only to one block. In the limit $p \to \infty$ we see from (16) that most multi-spin interaction terms are of an order proportional to $\log_2 p$. For this reason the MF theory of the infinite-range large-p Potts model is similar to the simplest SG model which has been shown to be equivalent to the 'random-energy' model. However, one can see from (16) that in the Potts model there is an additional finite number of multi-spin interactions of a finite order (compare with $\log_2 p$). Another difference between the models is that in the simplest SG model the number of interactions is proportional to $O(N^p/p!)$ where in the large-p Potts case the number of interactions is proportional to $O(pN^2)$. Qualitatively, we can *predict* from the mapping that in the case of dilute simplest spin glass, the smaller number of interactions does not effect the order function $q(x)$ but is responsible for the correlations among the energy levels as given in (13) and (14). This will now be shown qualitatively.

The Hamiltonian of the dilute simplest SG model is the same as in (4) in the limit of large-p multi-spin interactions. The $\{J_{i_1\ldots i_p}\}$ are randomly distributed according to the probability distribution

$$P(J_{i_1\ldots i_p}) = cP_1(J_{i_1\ldots i_p}) + (1-c)\delta(J_{i_1\ldots i_p}) \tag{17}$$

where P_1 is given by

$$P_1(J_{i_1\ldots i_p}) = (N^{p-1}c/J^2p!)\exp(-J^2_{i_1\ldots i_p}N^{p-1}c/J^2p!). \tag{18}$$

The distributions given by (17) and (18) indicate that the simplest SG model is diluted at random. The total number of remaining multi-spin interactions in the system after the dilution is $(N^p/p!)c$. Calculation of $P(E)$ for this model gives again

$$P(E) = (\pi NJ^2)^{-1/2}\exp(-E^2/J^2N) \tag{19}$$

where c satisfies the inequality

$$c > N^{-p+1+\varepsilon} \qquad \varepsilon > 0. \tag{20}$$

Calculation of $P_{12}(E_1, E_2)$ under the assumption of (20) gives

$$\begin{aligned} P_{12}(E_1, E_2) = {} & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathrm{d}Z_1\,\mathrm{d}Z_2}{(2\pi)^2}\exp(\mathrm{i}Z_1E_1 + \mathrm{i}Z_2E_2) \\ & \times \prod_{i_1<i_2\ldots<i_p}\Bigg[1 - c + c\exp\bigg(-\frac{J^2p!}{4N^{p-1}} \\ & \times (Z_1^2 + Z_2^2 + 2Z_1Z_2 s_{i_1}^1\ldots s_{i_p}^1 s_{i_1}^2\ldots s_{i_p}^2)\bigg)\Bigg]. \end{aligned} \tag{21}$$

From this it can be shown that $P(E_1, E_2) \neq P(E_1)P(E_2)$. However, calculation of the partition function for this model under the assumption of (20) gives again the step function solution given by (6) and (7) with $q = 1$ and $x = T/T_c$ for all $T < T_c$, where T_c is the same as in the undiluted case. Equations (17)–(21) prove that if the remaining multi-bonds is the dilute simplest SG model is greater than $N^{1+\varepsilon}$, $\varepsilon > 0$, the order function $q(x)$ is the same as in the undiluted case. The correlations between the energies are a function of the dilution of the system, the weight of the finite-order multi-spin interactions and the correlations among the interactions, as in the Potts case.

For a finite number of Potts states, the weight of the dominant-term multi-spin interactions of order $\log_2 p$ becomes smaller. Nevertheless, near the transition from the paramagnetic phase to the SG (or PG) phase, the finite p-states Potts model and the finite-p multi-spin interaction behave similarly, and $q(x)$ is the single-step function given in (6) and (7). At lower temperatures, each of the original pure SG (or PG) states splits into a hierarchical manifold of an infinite number of partially correlated states. The partial overlap between the states and the correlations among the energy levels can be different from one model to another.

We now generalise this mapping and claim that all discrete-spin Hamiltonian can be replaced by an equivalent Ising Hamiltonian by using the following rules.

(i) Each discrete spin, which can be in p different states, should be replaced by a block of $\log_2 p$ Ising spins. For the cases for which $p \neq 2^R$, $R = 2, 3, \ldots$, one should replace the discrete spin by $\log_2 p$ Ising spins, and add constraints in order to avoid the forbidden states.

(ii) The interactions among the blocks should be determined by the special symmetry of the original discrete-spin Hamiltonian. In order to obtain the explicit interactions among the blocks, one must represent the interaction between two (or more than two) discrete spins by a combination of Kronecker's δ-functions†. This combination of the δ-functions determines the explicit number and the form of the multi-spin interactions among the blocks in the equivalent Ising Hamiltonian.

The ability to replace each discrete-spin Hamiltonian by a compatible Ising Hamiltonian results from the symmetry of the system being determined by two independent variables:

(i) the discrete-spin representation;
(ii) the form of the interaction among the spins.

This means that if the spin representation is given, one can determine the symmetry of the system by the form of the interactions, and vice versa. This mapping can easily be extended to discrete-spin models with short-range interactions in any number of dimensions.

Another similarity between the multi-spin interactions and interactions between spins with more than two components is the nature of the spin's frustration. Given a loop with more than two negative bonds, one can then prove the following statement: the ground-state energy is unfrustrated when the number of spin components or the order of multi-spin interactions is greater than three. This result is in contra distinction to the well known result for the Ising system, where the frustration depends only on the sign of the total product of the bonds.

† In the worst case, for any possibility of an interaction between two blocks, one can write a special term, which is a combination of δ-functions.

Finally, we note as an application that in solving the Potts neural network [11] one can use the above mapping to explain the growth in the capacity of the system, relative to the Ising system, by using a multi-neuron synapse. the above mapping can also clarify what the weight is of the terms that are responsible for the breaking of the symmetry and may also be used to explain some results in random graphs with multi-edges.

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References

[1] Edwards S F and Anderson P W 1975 *J. Phys. F: Met. Phys.* **5** 965
[2] Mezard M, Parisi G, Sourlas N, Toulouse G and Virassora M 1984 *Phys. Rev. Lett.* **52** 1156
[3] Parisi G 1980 *J. Phys. A: Math. Gen.* **13** L115, L1101, 1887
[4] Gross D J, Kanter I and Sompolinsky H 1985 *Phys. Rev. Lett.* **55** 304
[5] Elderfield D and Sherrington D 1983 *J. Phys. C: Solid State Phys.* **16** L497
[6] Nishimori H and Stephen M J 1983 *Phys. Rev.* B **27** 5644
[7] Derrida B 1980 *Phys. Rev. Lett.* **45** 79; *Phys. Rev.* B **24** 2613
[8] Gross D J and Mezard M 1984 *Nucl. Phys.* B **240** 431
[9] Gardner E 1985 *Nucl. Phys.* B **257** 747
[10] Kirkpatrick S and Sherrington D 1978 *Phys. Rev.* B **17** 4384
[11] Kanter I and Sompolinsky H 1987 *Preprint*