Mean-Field Theory of Spin-Glasses with Finite Coordination Number

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The mean-field theory of dilute spin-glasses is studied in the limit where the average coordination number is finite. The zero-temperature phase diagram is calculated and the relationship between the spin-glass phase and the percolation transition is discussed. The present formalism is applicable also to graph optimization problems.

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The mean-field theory (MFT) of spin-glasses usually refers to an *infinite-range* system of N spins; each one of them is connected to the remaining $N-1$ spins.[1] In this paper, the MFT of *dilute* spin-glasses is studied. If, after dilution of the bonds, the average number of bonds per spin remains of $O(N)$, the dilution does not affect the behavior of the system. If, however, the average coordination number is *finite,* some new physics is expected to emerge. In particular there will be an interesting interplay between the statistical-mechanical frustration and the geometric connectivity fluctuations. Also, one might expect that some features of the diluted system are closer to the realm of the *short-range* spin-glass (SG) system.

Besides the relevance to the low-temperature properties of spin-glasses, the theory of dilute spin-glasses has important applications in graph optimization problems.[2] Some of these problems can be mapped into random, frustrated Ising models with highly diluted infinite-range interactions.[3] For these problems, mean-field theory should yield *exact results* in the thermodynamic limit.

The MFT of dilute SG's has been previously systematically studied only in the neighborhood of the transition temperature.[4] However, some of the interesting properties of the system are revealed at low temperatures. In fact, earlier treatments of the low-T phase failed to arrive at a consistent theory which incorporates the all-important frustration of the system.[5] We present here a MFT of dilute spin-glasses which is appropriate for all temperatures T and solve it in the limit of $T\to 0$. The phase diagram and the properties of the various low-T phases are discussed.

We consider an Ising system described by the Hamiltonian

$$H=-\frac{1}{2}\sum_{i\neq j}J_{ij}S_iS_j, \tag{1}$$

where $S_i=\pm 1$ $(i=1,\ldots,N)$, and the J_{ij}'s are *infinite-ranged* random interactions. Their probability distribution is

$$p(J_{ij})=(1-c/N)\delta(J_{ij})+(c/N)f(J_{ij}). \tag{2}$$

It describes a network of bonds which is highly diluted: The average coordination number of each spin is c which is taken to be on the order of 1. The distribution of the surviving bonds is given by $f(J_{ij})$ which is normalized to unity. Because the average number of bonds is $cN/2$ (and not $N^2/2$) the scale of J_{ij} must be on the order of 1 to achieve the appropriate thermodynamic limit.

Since there are no length scales in the problem, a mean-field theory is expected to give an exact description of the system in the thermodynamic limit $(N\to\infty)$. Indeed, using the replica method, Viana and Bray[4] have shown that the average free energy per spin at temperature $T=\beta^{-1}$ can be expressed as

$$\beta f=\frac{c}{n}\left\{\frac{a_1}{2}\sum_{\alpha}Q_\alpha^2+\frac{a_2}{2}\sum_{\alpha<\beta}Q_{\alpha\beta}^2+\frac{a_3}{2}\sum_{\alpha<\beta<\gamma}Q_{\alpha\beta\gamma}^2+\ldots\right\}-\frac{1}{n}\ln \mathrm{Tr}_{S^\alpha}\exp(-\beta\bar{H})-\frac{c}{n}\ln\int dJ\,f(J)\cosh^n(\beta J), \tag{3}$$

$$\beta\bar{H}=-c\left[a_1\sum_\alpha Q_\alpha S^\alpha+a_2\sum_{\alpha<\beta}Q_{\alpha\beta}S^\alpha S^\beta+a_3\sum_{\alpha<\beta<\gamma}Q_{\alpha\beta\gamma}S^\alpha S^\beta S^\gamma+\ldots\right]. \tag{4}$$

The indices $\alpha,\beta,\gamma,\ldots$ run from 1 to n. The variables S^α represent n spins *at the same site*. The constants a_k are

$$a_k=\int_{-\infty}^{\infty} dJ\, f(J)\tanh^k(\beta J). \quad (5)$$

The physical free energy is derived by minimization with respect to $(Q_\alpha,Q_{\alpha\beta},Q_{\alpha\beta\gamma},\ldots)$ and taking of the limit $n\rightarrow 0$.

The new feature of Eq. (3) is the appearance of a large number of order parameters, whereas in the MFT of *undiluted* SG's [the Sherrington-Kirkpatrick (SK) model[1]] only Q_α and $Q_{\alpha\beta}$ appear. The present model has been studied in Ref. 4 near the transition temperatures. There, to leading order, one can neglect all but a few of the order parameters which results, not surprisingly, in a behavior qualitatively similar to that of the SK model. Here we focus on the low-temperature regime where all order parameters are of the same magnitude. The study of the low-T limit is further complicated by the $n\rightarrow 0$ limit. Earlier attempts[5] have shown that in frustrated systems this limit has to be taken *before* the $T\rightarrow 0$ limit.

In this paper we solve the problem within the framework of replica-symmetric theory. The order parameters are assumed to be independent of the replica indices, i.e., $Q_\alpha=Q_1$, $Q_{\alpha\beta}=Q_2$, $Q_{\alpha\beta\gamma}=Q_3$, etc., for all replica indices. The quantities Q_k are simply the moments of the local magnetizations, $Q_k=\langle\langle m_i^k\rangle\rangle$, where $m_i\equiv\langle S_i\rangle_T$, where $\langle\ldots\rangle_T$ is a thermal average, and $\langle\langle\ldots\rangle\rangle$ stands for an average over the J_{ij}. On the assumption of this structure, the limit $n\rightarrow 0$ of Eq. (3) can be taken explicitly, yielding a free energy which is a function of all order parameters Q_k. Instead of dealing directly with an infinite number of order parameters, it is most useful to consider the probability distribution of the local fields defined by $h_i\equiv\tanh^{-1}\langle S_i\rangle_T$. Note that h_i is not equivalent to the *exchange* field $\sum_j J_{ij}m_j$. As $T\rightarrow 0$, $T|h_i|$ is the minimum energy cost for changing the ith spin from its ground state by an arbitrary excitation which involves the flipping of a finite number of spins. The free energy can be expressed as a functional of the averaged, local-field distribution, $P(h)$. Extremizing this free energy, we have derived the following self-consistent equation for the "order function" $P(h)$:

$$P(h)=e^{-c}\int_{-\infty}^{\infty}\frac{dy}{2\pi}\exp\left[-iyh+c\int_{-\infty}^{\infty}dJ\,f(J)\int_{-\infty}^{\infty}P(x)dx\exp\{iy\tanh^{-1}[\tanh(\beta J)\tanh x]\}\right]. \quad (6)$$

Near the transition temperature the local fields are small, and hence one can expand the exponent in Eq. (6) in powers of $m(x)=\tanh x$. This leads to self-consistent equations for the lowest moments of m, which recover the results of Ref. 4. For general T, and arbitrary bond distribution $f(J)$, Eq. (6) can be solved numerically. Here we specialize to the particularly simple case of the discrete bond distribution,

$$f(J)=a\delta(J-1)+(1-a)\delta(J+1), \quad (7)$$

in the limit of $T\rightarrow 0$. Since in this case the excitation energies are integers, $P(h)$ must have, at zero T, the following form:

$$P(h)=(1-Q)\delta(h)+\sum_{l=1}^{\infty}P_l^+\delta(h-\beta l)+\sum_{l=1}^{\infty}P_l^-\delta(h+\beta l). \quad (8)$$

With the definitions $P^\pm=\sum_l P_l^\pm$, it is evident that $P^++P^-=Q$, $P^+-P^-=m$, where Q is the total fraction of frozen spins, and m is the net magnetization (per spin) of the frozen spins. In taking the $T\rightarrow 0$ limit of Eq. (6) we note that $\lim_{T\rightarrow 0}|\tanh^{-1}[\tanh(\beta J)\tanh x]|$ is equal to $\beta|J|$ if $|x|\geq\beta|J|$ and to $|x|$ if $|x|\leq\beta|J|$. Substituting Eq. (8) into the right-hand side of Eq. (6), one then finds

$$P(h)=e^{-c}\int_{-\infty}^{\infty}\frac{dy}{2\pi}\exp[-iyh+cQ(x_+e^{+iy\beta}+x_-e^{-iy\beta})], \quad (9)$$

where

$$x_\pm=\tfrac{1}{2}\pm(m/Q)(a-\tfrac{1}{2}). \quad (10)$$

Expanding the integrand of Eq. (9) in powers of cQ and integrating over y, one obtains a series of δ functions in h which confirms the consistency of the *Ansatz* (8). Furthermore, summing all contributions to $\delta(h)$, and calculating the difference between the contributions with positive h and contributions with negative h, one obtains the following equations for Q and m:

$$1-Q=e^{-cQ}I_0(2cQ(x_+x_-)^{1/2}), \quad (11)$$

$$m=cQe^{-cQ}\int_{x_-}^{x_+}dt\{I_0(2cQ[t(1-t)]^{1/2})+[4t(1-t)]^{-1/2}I_1(2cQ[t(1-t)]^{1/2})\}, \quad (12)$$

where $I_\nu(x)$ are modified Bessel functions. Evaluating the free energy at low T, we find for the *ground-state energy per spin, E*, the following expression:

$$E=-\tfrac{1}{2}c(1-Q)^2+c(a-\tfrac{1}{2})m^2-T\int_{-\infty}^{\infty}dh\,P(h)|h|. \quad (13)$$

We now discuss a few consequences:

(i) Dilute ferromagnet, $a=1$.—In this case $P^-=0$, and Eqs. (11) and (12) reduce to $Q=m=P$, where P is the order parameter of the intinite-range percolation,[2] satisfying

$$1-P=e^{-cP}. \tag{14}$$

It is nonzero above the percolation threshold $c=1$ and approaches unity with $c\to\infty$ as $P\sim 1-e^{-c}$. In addition, Eq. (9) yields the interesting result that $P_l^+=(cP)^l e^{-cP}/l!$. Note that by its definition, Eq. (8), P_l^+ is (in this case of $a=1$) the average concentration of spins that can be disconnected from the infinite cluster by cutting only l bonds. For instance, $NP_1^+=NcPe^{-cP}$ is the average number of sites on the infinite cluster which do not belong to its "backbone." Equation (13) yields $E=-\frac{1}{2}c$ for all c which is just the average number of bonds per spin as expected.

(ii) Spin-glass phase, $a<1$.—For all $a<1$, there is a range of $c>1$ where a spin-glass phase exists, characterized by $m=0$, $Q\neq 0$. In this phase Eq. (11) reduces to

$$1-Q=e^{-cQ}I_0(cQ). \tag{15}$$

It is nonzero above $c=1$ with $Q\sim\frac{4}{3}(c-1)$, near $c\sim 1^+$. As $c\to\infty$, Q approaches unity only as a power law, $1-Q\sim(2\pi c)^{-1/2}$. Note that for all $c>1$, Q is *less* than P as shown in the inset of Fig. 1. *The difference $P-Q$ represents the average concentration of frustrated spins,* i.e., the concentration of spins on the infinite cluster which can be flipped at $T=0$ by an excitation with zero energy. The ratio $(P-Q)/P$ has the value $\frac{1}{3}$ in the limit of $c\to 1^+$, implying that at the percolation threshold one-third of the spins on the percolating cluster are frustrated.

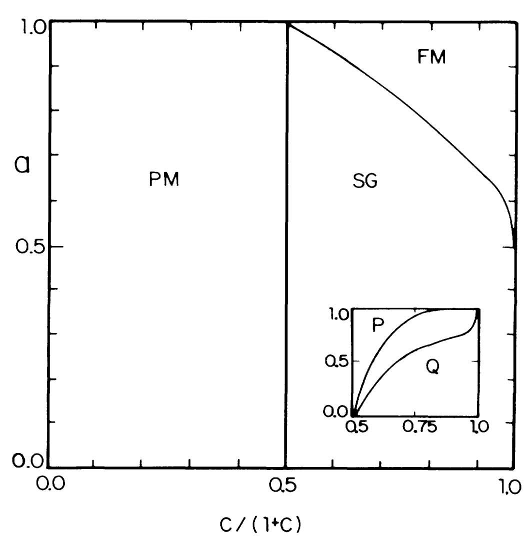

FIG. 1. The zero-T phase diagram: a is defined in Eq. (7); PM, FM, and SG stand for paramagnetic, ferromagnetic, and spin-glass phases. Inset: The percolation order parameter, P, Eq. (14), and the SG order parameter, Q, Eq. (15), as functions of $c/(1+c)$ in the SG phase.

Another aspect of the frustration is given by the ground-state energy. Evaluating Eq. (13) in the SG phase, we obtain

$$E_{\rm SG}=-\tfrac{1}{2}c(1-Q)^2-cQe^{-cQ}[I_0(cQ)+I_1(cQ)]. \tag{16}$$

This energy is always greater than $-\frac{1}{2}c$. In fact, the quantity $N(E_{\rm SG}+\frac{1}{2}c)/2$ is the total number of *unsatisifed bonds* in the SG phase. Near the transition, $E_{\rm SG}+\frac{1}{2}c\simeq\frac{1}{8}(c-1)^3$.

The SG zero-T critical behavior near $c=1$ can be understood in terms of the geometrical properties of the percolating cluster in many dimensions.[6] According to the "nodes and links" model,[6] the typical loops on the infinite cluster are of linear size ξ, which is the percolation correlation length $\xi\sim(p-p_c)^{-\nu}$. The total number of loops in a system with linear size L is proportional to $(L/\xi)^d=N(p-p_c)^{\nu d}$, which implies that the number of loops *per site* at the upper critical dimension, $d=6$, is proportional to $(p-p_c)^3$. A finite fraction of the loops are frustrated and since each frustrated loop contains roughly one frustrated bond, the frustration energy must be proportional to $(p-p_c)^3$. Note that for many dimensions the *finite* clusters have a treelike structure with no loops, and hence $E_{\rm SG}=-\frac{1}{2}c$ below $c=1$. Most of the spins which are on a frustrated loop are frustrated since they can be flipped by the movement of the frustrated bond along the loop. Furthermore, the flip of a spin will cause the flipping of all the spins which lie on dangling ends attached to it; hence most of the dangling ends attached to the frustrated loops are frustrated. The total mass of these dangling ends is proportional to PN and therefore a finite fraction of the spins on the percolation cluster are frustrated even at $c=1^+$. Note that Eqs. (15) and (16) are independent of the parameter a, meaning that the properties of the SG phase at $T=0$ are determined entirely by the geometry of the loops on the infinite cluster and are therefore independent of the relative concentration of negative bonds. The *extent* of this phase does depend on a as discribed below.

(iii) Ferromagnetic phase, $\frac{1}{2}<a<1$.—Expanding Eq. (12) in small m, one finds a transition from a SG phase to a ferromagnetic phase which occurs at the critical value of c given by the equation

$$1=(2a-1)ce^{-cQ}[I_0(cQ)+I_1(cQ)], \tag{17}$$

Q being the SG order parameter. When $0\le a\le\frac{1}{2}$, Eq. (17) does not have a solution, implying that when the concentration of negative bonds is higher than the positive ones the SG phase given by Eqs. (15) and (16) exists at $T=0$ for all c. For $a\ge\frac{1}{2}$, a ferromagnetic phase appears above the critical c, and is characterized by $q>m>0$.

The full phase diagram at $T=0$ is shown in Fig. 1. Note that although the *undiluted,* infinite-range antiferromagnet remains paramagnetic even at zero temperature, the dilute antiferromagnet (the present model with $a=0$) freezes into a SG state for *all* $c>1$. Thus, any arbitrarily weak dilution of the infinite-range antiferromagnet pins a large fraction of the frustrated bonds and causes a freezing of the system at low T.

Many of the above qualitative results, including the general form of the phase diagram, are valid also for *bond* distributions other than Eq. (8). Perhaps the most important difference is associated with the value of the SG order parameter Q. By analysis of the contribution of Eq. (6) to $\delta(h)$ it is straightforward to see that if the bond distribution $f(J_{ij})$ is *continuous,* the equation for Q is just $1-Q=\exp(-cQ)$, which is the same as that of the percolation order parameter P, Eq. (14). This is indeed expected. In the continuous case the probability that local fields on the percolating cluster vanish is zero, and hence all the spins on the finite cluster are frozen at $T=0$. It should also be noted that the critical properties of the zero-T SG transition at $c=1$ depend on the form of the bond distribution at the origin, similar to a one-dimensional SG. If $f(J\rightarrow 0)\sim J^n$ then, by extending the previous arguments, one expects that the energy singularity is $E_{SG}\sim(c-1)^{3+1/(n+1)}$.[7]

The present mean-field solution recovers exactly the results of the replica-symmetric MFT of the SK model,[1] in the limit of $c\rightarrow\infty$, except for the scaling of the exchange by a factor $1/\sqrt{c}$. For instance, as $c\rightarrow\infty$, Eq. (16) yields $E_{SG}/\sqrt{c}=-(2/\pi)^{1/2}$, and at the SG-ferromagnetic transition line, Eq. (17) yields $\sqrt{c}\times(a-\frac{1}{2})=\sqrt{\pi/2}$ which agrees with the SK results. The replica-symmetric theory is unstable in the SK limit and it might be unstable at all values of $c>1$. Indeed, Viana and Bray[4] found an instability to the breaking of replica symmetry near the SG transition temperature for all $c>1$. It would be very interesting to understand the nature of this symmetry breaking in the present model particularly at $T=0$. The present approach can be used to study graph partitioning problems. This application which involves extending our mean-field theory to incorporate the effects of *external* fields is currently being investigated. Details of the calculations as well as the derivation of Eq. (6) will be given elsewhere.

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Note added.—Order functions which are related to our $P(h)$, Eq. (6), have recently been introduced by DeDominicis and Mottishaw[8] and by Bray.[8]

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