Mean-Field Theory of the Potts Glass

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Randomly interacting p-state Potts spins may freeze into a Potts-glass phase in which the Potts symmetry is unbroken, on the average. The mean-field theory of this phase transition is presented. Unlike the spin-glass case, there exist two distinct Potts-glass phases that differ in the nature of the correlations among the many degenerate ground states of the system. For $p>4$, the transition from the disordered phase is unusual: The freezing occurs discontinuously but without latent heat. Similar results hold for mean-field quadrupolar-glass models.

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Magnetic systems with random competing interactions often condense at low temperature into a spin-glass (SG) phase. The mean-field theory based on the Edwards-Anderson model[1] reveals that the phase transition to the SG phase, when it occurs, is a continuous transition similar to an ordinary second-order phase transition.[2] However, the SG phase itself is very unusual. It consists of many degenerate partially overlapping pure states, which exhibit a simple hierarchical organization.[3] It can be characterized by an order parameter $q(x)$, whose inverse $x(q)$ determines the probability that two pure states have an overlap (fraction of common spins) less than or equal to q. Recently there has been a growing tendency to apply concepts of the SG theory to various nonmagnetic random "orientational glasses" such as K(Br,CN) mixed crystals,[4] electric dipole glasses,[5] and mixed ortho-para hydrogen crystals.[6,7] Some of these systems do not possess reflection or rotation symmetries characteristic of spin systems and are more appropriately described by Hamiltonians of randomly interacting Potts variables (Potts glass) or quadrupoles (quadrupolar glass). Recent investigations of the mean-field theory of these types of models failed to find a self-consistent stable low-temperature phase using the replica method.[8-11] This raised doubts on the applicability of the current SG theory to nonmagnetic randomly frustrated systems.

In this Letter we present the results of a mean-field theory of the Potts glass. This corresponds to physical situations where (because of strong crystal fields) the local degrees of freedom are restricted to a finite number of "orientations" or states. At the end, we will discuss briefly quadrupolar glass models, which are appropriate for continuous degrees of freedom. We envisage a system of N p-state Potts variables $\sigma(i)=0,1,2,\ldots,p-1$. The random pair interactions are *infinite ranged* and given by the following Hamiltonian[8,9]:

$$H=\frac{-p}{2}\sum_{i\neq j}J_{ij}\delta_{\sigma(i),\sigma(j)}. \tag{1}$$

The couplings J_{ij} are quenched Gaussian-distributed variables with mean $[J_{ij}]=J_0/N$, variance $[J_{ij}^2]=J^2/N$. We denote by Potts glass (PG) a state of the system in which the local $\sigma(i)$ freeze in particular states but *on the average* all the states are equally occupied. More explicitly, *all ensemble-averaged quantities in the Potts-glass phase are invariant under global permutations of the states.* In particular, the "ferromagnetic" order parameters $m_r=p[\langle\delta_{\sigma(i),r}\rangle]-1$ are zero. Here $\langle\ldots\rangle$ denotes thermal average in a particular realization of J_{ij} and $[\ldots]$ refers to "ensemble" average over the J_{ij}. The PG order parameter

$$Q_{rs}=[(p\langle\delta_{\sigma(i),r}\rangle-1)(p\langle\delta_{\sigma(i),s}\rangle-1)],$$

which is the analog of the Edwards-Anderson order parameter, has the symmetry $Q_{rs}=\bar{q}(p\delta_{r,s}-1)$, $0\leq|\bar{q}|\leq 1$.

For such a phase to be stable in the case of (1), an appropriate nonzero value of J_0 must be chosen.[8] Related models are the random chiral model introduced by Nishimori and Stephen,[10] with each $J_{ij}^{(r)}$ an independent random variable, and the randomly anisotropic Potts Hamiltonian. In both these models the Potts symmetry is *not* an exact symmetry of each random realization. However, the low-temperature phase in both cases is a Potts glass if the mean of the couplings is zero. *Although the three models are very different, the PG phases of these models, whenever they are stable, are identical, within mean-field theory.* We first summarize our main results:

(1) Application of Parisi's replica theory[3,12] yields a consistent, stable PG phase for all values of p. This phase consists, as in the SG case, of an infinite number of pure degenerate PG states, which appear below a critical temperature, T_c.

(2) Unlike the SG case, the order function $q(x)$ for $p \geqslant 3$ is *discontinuous.* In particular, just below T_c, the system condenses into a phase (PG1) in which $q(x)$ is a single step function [see Fig. 1(a)]. This implies that the phase space splits into an infinite number of distinct pure PG states which *do not overlap.* Near T_c this phase is stable: All eigenvalues of fluctuations about it are *positive.* This is to be contrasted with the enormous marginality[13] of the SG phase.

(3) The above PG phase becomes unstable at a lower temperature T_2. At T_2, the system undergoes a phase transition into a second PG phase (PG2). In PG2, *each* of the original pure PG states splits into a hierarchical manifold[3] of an infinite number of partially correlated states. This is represented by an order function of the form shown in Fig. 1(b).

(4) In three-state and four-state systems both phase transitions are *continuous.* On the other hand, for $p > 4$ the transition at T_c from the disordered state to the PG1 state is *discontinuous.* Note that in the infinite-ranged *uniform* Potts model [Eq. (1) with $J=0$, $J_0 \neq 0$], the transition is first-order for all $p > 2$.[14]

(5) The discontinuous PG transition, in the case of $p > 4$, has unusual properties. The "single-valley" order parameter, q, jumps discontinuously from zero at T_c implying the discontinuous freezing of the local degrees of freedom in each of the valleys. This would be reflected by a discontinuity in the *ac response* to an external source.[15] Also, all "perturbative" fluctuations about the disordered state remain finite at T_c, which implies that all "PG susceptibilities" remain finite as T approaches T_c from above. Thus, T_c is not an ordinary critical point. Neither is the transition an ordinary first-order one. The free energy depends on the order parameters only via integrals such as $\int_0^1 q^2(x)\,dx = q^2(1-\bar{x})$ where $\bar{x}$ is the break point of the step function $q(x)$ [see Fig. 1(a)]. But $1-\bar{x}(T)$ vanishes at T_c as $T_c - T$. Consequently, only *second derivatives* of the free energy with respect to temperature and *static* external fields are discontinuous at T_c, and in particular, *there is no latent heat at the transition.*

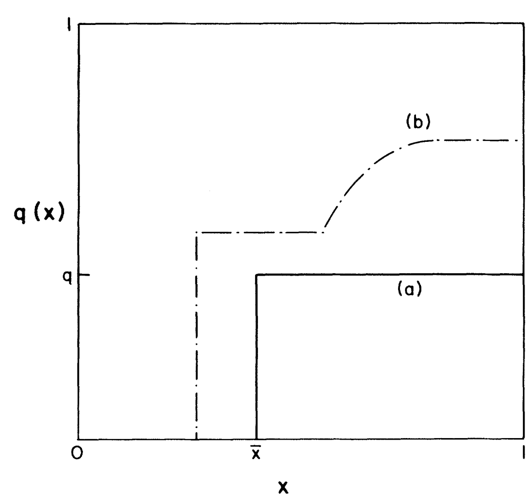

FIG. 1. Schematic plots of the shape of the order function $q(x)$ in the two different PG phases. (a) The first PG phase, $T_2 < T < T_c$; (b) the second PG phase, $0 < T < T_2$.

We now discuss in more detail the nature of this unusual discontinuous transition, at $p > 4$. From thermodynamic point of view, the vanishing of latent heat is associated with the fact that the PG free-energy branch lies *above* the analytical continuation of the disordered, paramagnetic (PM) state. This means that a finite latent heat cannot appear at T_c since it would necessarily be negative. The "anomaly" that the low-T free energy is higher than the analytical continuation of the PM one is common to all known SG transitions in mean-field theory. However, usually, the PM phase is unstable at the temperature where the SG phase appears, whereas here, the naive expansion in fluctuations about the PM phase does not show a singularity at T_c, for $p > 4$. Nevertheless, the transition from the PM to the PG phase must occur at T_c, since it is there that the free energies of the two phases are equal. This conclusion implies that the analytically continued PM branch below T_c does not represent a physical state, not only in the thermodynamic sense but also in the sense of a metastable state. This should be manifested as a nonperturbative singularity in the finite-N corrections about this phase. Thus the notion of two metastable phases between spinodal points of ordinary first-order transitions does not seem to apply here. Although the nature of the instability of the PM state as T decreases below T_c is not yet clear to us, it is probably related to the fact that at T_c, the free-energy barrier between the two coexisting phases, the PG and the PM ones, at T_c is not *proportional to N, but to* $\sqrt{N}$. A similar reduction seems to occur in the barrier between the coexisting SG ground states *below* T_c,[16] and is also consistent with the vanishing of surface tensions[17] or static stiffness constants[18] in short range spin-glass systems.

Further insight into that transition comes from the scaling behavior of the free energy f at T_c. We find that the dependence of the singular part of f on T and external *static* sources obeys ordinary scaling laws with the critical exponents

$$\beta = 1, \quad \gamma = 0, \quad \alpha = 0, \quad \delta = 1. \qquad (2)$$

If Eq. (2) holds also for short-range systems in some range of dimensionality d, then the hyperscaling law $\nu d = 2 - \alpha$ suggests that the thermal exponent which determines the rounding of the transition due to finite size obeys $1/\nu = d/2$. This should be compared with the relation $1/\nu = d$ which holds in ordinary first-order transitions.[19] The difference is again indicative of the above-mentioned square root of volume effect characterizing the random system.

The above results have been derived from the replica mean-field theory[12] in which the order parameters are

$$q_{\alpha\beta} = [\langle \delta_{\sigma^\alpha,\sigma^\beta} \rangle] - 1/p. \qquad (3)$$

The variables σ^α and σ^β are the local Potts spins of two copies of the system, and we have used the Potts invariance, similar to Eq. (3). If replica symmetry is unbroken, then $q_{\alpha\beta} = \bar{q}$. We have studied the full replica theory of the Potts models, where in the case of (1) an appropriate negative J_0 was chosen.[8] However, in order to understand the transition from the disordered state to the PG1 phase it is sufficient to consider a simplified Landau free energy which leads to qualitatively correct behavior near T_c. This free energy is of the form

$$\frac{-f}{(p-1)} = \frac{1}{2n}\left\{ t\,\mathrm{Tr}\,q^2 + \frac{1}{3}\mathrm{Tr}\,q^3 + \frac{p-2}{6}\sum_{\alpha\neq\beta} q^3_{\alpha\beta} - \frac{y}{12}\sum_{\alpha\neq\beta} q^4_{\alpha\beta} \right\}, \qquad (4)$$

where $t = 1 - T/J$, the Tr symbols refer to the replica indices, and n, the number of the replicas, approaches zero. Equation (4) differs from the Ising spin-glass case ($p=2$) in two important aspects: (i) The coefficient of $\sum q^3_{\alpha\beta}$ vanishes in the case of $p=2$ (Ref. 12) by the reflection symmetry of the spins, and (ii) the coefficient y of the quartic term is *negative* for $p=2$. In fact, we find that y becomes positive for $p > p^*$, where $p^* \sim 2.8$.

The positive sign of y for $p > p^*$ is responsible for the lack of a solution to the saddle-point equations $\partial f/\partial q_{\alpha\beta} = 0$ which has a continuous Parisi function $q(x)$. The *only Parisi-type solution,* besides the unstable replica-symmetric one, *is of the form of Fig. 1(a),* with $q \simeq 2t/(4-p)$, $\bar{x} \simeq (p-2)/2$, which is valid for $t \gtrsim 0$ and $p^* < p < 4$. All eigenvalues of fluctuations about this saddle point are *positive* (in this temperature range).

The above solution ceases to exist above $p=4$, and in general when the coefficient of the second cubic term in (4) becomes larger than the first one. For $p > 4$, one must consider a discontinuous jump of q as a function of temperature. Since q is no longer a small parameter, the model (4) is no longer valid in general for $p > 4$. However, one can still use it in the limit of $\epsilon = p - 4 \to 0$. Using ϵ as a small parameter one obtains a PG phase with broken replica symmetry of the above form, appearing below $T_c/J - 1 \propto \epsilon^2$, with $q(T_c) \propto \epsilon$ and $\bar{x}(T \to T_c^-) \to 1$. Fluctuations around both this saddle point and the paramagnetic state, $q=0$, are all finite near T_c.

The result $\bar{x}(T \to T_c) \to 1$ is a crucial property of the discontinuous transition as was discussed above. It holds order by order in expansion in powers of ϵ. It also holds in the limit of large p. Solving the full mean-field theory of the PG phase in the limit of $p \to \infty$ we find a discontinuous transition from the PM phase at the temperature where its entropy vanishes, which is

$$T_c = \tfrac{1}{2} J (p/\ln p)^{1/2}. \qquad (5)$$

Below T_c, a PG phase appears with the order parameters $q(T) = 1$, $\bar{x}(T) = T/T_c$. The PG free energy is constant and equals the ground-state energy $E_0 = -J(p\ln p)^{1/2}$. This phase transition is identical to that of a "random-energy" model consisting of p^N states, whose energies are independent random variables.[20,21] The equivalence of the large-p Potts model to the random-energy model is due to the entropic suppression of overlaps of different states, in the limit of $p \to \infty$.

Interestingly, the leading finite-size correction to the high-temperature free energy per site in the random-energy model has a singularity at T_c, of the form[20]

$$\delta f \propto \frac{1}{N^{3/2}\tau} \exp(-AN\tau^2), \quad \tau = T - T_c > 0. \qquad (6)$$

We conjecture that the finite-size corrections of the PM free energy of the above Potts models have a singularity at T_c similar to (6) for all $p > 4$. Such a nonperturbative singularity would signal the instability of the paramagnetic state at T_c where perturbative fluctuations are still finite. Note also that both Eq. (6) and the finite-size correction of the random-energy model, *below* T_c,[20] are consistent with a finite-size scaling of the free energy per site, $f(\tau,N) = \tau^{2-\alpha} g(\tau^y N)$, with $\alpha = 0$ and $y = \nu d = 2$, which agrees with the above scaling.

In the large-p limit, the PG1 phase persists down to $T=0$. However, for any finite $p > 2$ this phase has a negative entropy at $T=0$. In fact, it becomes unstable at a temperature $T_2 < T_c$ to inducing replica-symmetry breaking in the region $\bar{x} < x < 1$. This gives rise to the PG2 phase in which $q(x)$ is continuous for some range of x. This phase is marginal and has a zero entropy at $T=0$. It is difficult in general to study analytically the properties of the PG2 phase. However, one can consider an expansion of the free energy similar to Eq. (4) but with the addition of a fifth-order term, in the limit of $p - p^* \to 0$. Then, one sees explicitly the instability of the PG1 phase at T_2 and the emergence of a new phase, with an order function of the form of Fig. 1(b). This second transition is continuous (as a function of T), even for $p \gtrsim 4$.

Finally, we briefly discuss the mean-field theory of the quadrupolar glass (QG). We have studied infinite-range Hamiltonians of the type

$$H = -\frac{1}{2}\sum_{i \neq j}\sum_{\substack{\mu\nu \\ \rho\sigma}}^{m} J_{ij}^{\mu\nu,\rho\sigma} f^{\mu\nu}(i) f^{\rho\sigma}(j), \tag{7}$$

where $f^{\mu\nu}(i) = S^{\mu}(i) S^{\nu}(i) - \delta^{\mu\nu}$, $\mathbf{S}(i)$ is a classical m-component vector with fixed length $|\mathbf{S}|^2 = m$, and $J_{ij}^{\mu\nu,\rho\sigma}$ are random variables. The properties of the model and, in particular, the possible appearance of long-range order depend on the symmetry of the $J_{ij}^{\mu\nu,\rho\sigma}$ as well as on the value of their mean,[11] J_0/N. These questions will be discussed elsewhere. Here we concentrate on the *isotropic* QG phase, described by a single order parameter

$$q = \frac{1}{m^2}\sum_{\mu\nu}[\langle f^{\mu\nu}(i)\rangle^2].$$

The replica theory of this phase leads, for $m > 2$, to the same qualitative results as in the PG case. Near $T = J$ the system may be described by a Landau theory, qualitatively similar to (4). The lack of reflection symmetry (of the variables $f^{\mu\nu}$) gives rise to the second cubic term (for $m > 2$). The coefficient of the important quadratic term changes sign at $m > m^*$, $m^* \cong 2.67$. Thus, for $m > m^*$ the system has QG phases of the forms of Fig. 1. The transition from the high-temperature phase to the QG phase becomes discontinuous[11] at $m > m_c \sim 3.4$. Expansion in $m - m_c$ as well as an exact solution of the model in the large-m limit confirm that the properties of that discontinuous solution are similar to those of the $p > 4$ Potts glass. Note, however that unlike the PG, the quadrupolar system does not freeze *completely* at finite $T < T_c$ even in the limit of large m.

Finally we note that a discontinuous transition similar to that described above is found also in infinite-ranged SG with p-spin interactions ($p > 2$).[20,21] The fact that such a transition exists in a wide variety of models suggests that it is indeed the generic discontinuous transition of spin-glass-like systems in mean-field theory. An important open question is whether this transition exists also in short-range systems in some range of dimensionality. More details of the theory as well as a discussion on its relevance to experiments will be presented elsewhere.

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[1] S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).

[2] For a discussion of the scaling properties of the SG transition see D. S. Fisher and H. Sompolinsky, Phys. Rev. Lett. **54**, 1063 (1985).

[3] M. Mezard, G. Parisi, N. Sourlas, G. Toulouse, and M. Virasoro, Phys. Rev. Lett. **52**, 1156 (1984).

[4] K. H. Michel and J. M. Rowe, Phys. Rev. B **22**, 1417 (1980); S. Bhattacharya, S. R. Nagel, L. Fleishman, and S. Susman, Phys. Rev. Lett. **48**, 1267 (1982); A. Loide, K. Knorr, R. Feile, and J. K. Kjems, Phys. Rev. Lett. **51**, 1054 (1983); J. Ihm, Phys. Rev. B **31**, 1674 (1985).

[5] U. T. Hochli, Phys. Rev. Lett. **48**, 1494 (1982); E. Courtens, Phys. Rev. Lett. **52**, 69 (1984).

[6] N. S. Sullivan, M. Devoret, and D. Esteve, Phys. Rev. B **30**, 4935 (1984).

[7] A. B. Harris and H. Meyer, Can. J. Phys. (to be published).

[8] D. Elderfield and D. Sherrington, J. Phys. C **16**, L497 (1983); L971 (1983); L1169 (1983); D. Elderfield, J. Phys. A **17**, L517 (1984).

[9] E. J. S. Lage and A. Erzan, J. Phys. C **16**, L873 (1983); E. J. S. Lage and J. M. Nunes da Silva, J. Phys. C **17**, L593 (1984).

[10] H. Nishimori and M. J. Stephen, Phys. Rev. B **27**, 5644 (1983).

[11] P. Goldbart and D. Sherrington, to be published.

[12] G. Parisi, J. Phys. A **13**, L115, L1101, 1887 (1980).

[13] H. Sompolinsky and A. Zippelius, Phys. Rev. Lett. **50**, 1297 (1983); C. de Dominicis and I. Kondor, Phys. Rev. B **27**, 606 (1983).

[14] F. Y. Wu, Rev. Mod. Phys. **54**, 235 (1982).

[15] H. Sompolinsky, Phys. Rev. Lett. **47**, 935 (1981).

[16] N. D. Mackenzie and A. P. Young, Phys. Rev. Lett. **49**, 301 (1982).

[17] A. J. Bray and M. A. Moore, J. Phys. C **17**, L463 (1984).

[18] H. Sompolinsky, G. Kotliar, and A. Zippelius, Phys. Rev. Lett. **52**, 392 (1984).

[19] M. E. Fisher and A. N. Berker, Phys. Rev. B **26**, 2507 (1982).

[20] B. Derrida, Phys. Rev. B **24**, 2613 (1981).

[21] D. J. Gross and M. Mezard, Nucl. Phys. **B240**, 431 (1984).